Adelic approximation of generalized integral points

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Conventions

For simplicity, we work over $\mathbb{Q}$ and $\mathbb{Z}$, but the results work more generally over number fields and function fields of curves over any field. We denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

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Complete $\mathbb{Q}$ with $|\cdot|_p$ to get $\mathbb{Q}_p$, which is a locally compact field.
The closure of $\mathbb{Z}$ is the ring

$$\mathbb{Z}_p = \{a \in \mathbb{Q}_p \colon |a|_p \leq 1\}.$$

Strong approximation

By definition, $\mathbb{Z}$ dense in $\mathbb{Z}_p$, but by the Chinese remainder theorem we even have

Strong approximation theorem (weak form)

The diagonal embedding

$$\mathbb{Z} \hookrightarrow \prod_{p \text{ prime}} \mathbb{Z}_p$$

has dense image.

Generalizations of strong approximation

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Points 1 & 2 have been studied extensively, but very little is known about point 3.

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Toric varieties

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1. Products of projective spaces.
2. Hirzebruch surfaces H_d: which are a quotient like $\mathbb{P}^1 \times \mathbb{P}^1$, with instead the relation

$$(x_1 : x_2 : x_3 : x_4) = (\lambda x_1 : \mu x_2 : \lambda x_3 : \lambda^d \mu x_4).$$

Many types of points

Let X be a compact variety over $\mathbb{Q}$ with model $\mathcal{X}$ over $\mathbb{Z}$ and choose divisors $D_1, \ldots, D_n$ on X with closure $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\mathcal{X}$. There are a lot of special subsets of rational points defined relative to these,

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- integral points,
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We introduce W-points as a common framework for these points.

Multiplicity map

For a prime p and $P \in (X \setminus D_i)(\mathbb{Q})$ we define the **multiplicity** at a divisor $\mathcal{D}_i$ as the largest integer $N = n_p(P, \mathcal{D}_i)$ such that P mod p^N lies in $\mathcal{D}_i(\mathbb{Z}/p^N\mathbb{Z})$. If $P \in D_i(\mathbb{Q})$ we set $n_p(P, \mathcal{D}_i) = \infty$. Using this we define the multiplicity map

$$\mathrm{mult}_p \colon X(\mathbb{Q}_p) \to \overline{\mathbb{N}}^n$$

$$P \mapsto (n_p(P, \mathcal{D}_1), \ldots, n_p(P, \mathcal{D}_n)).$$

$\mathcal{W}$-points

Given $\mathfrak{W} \subset \overline{\mathbb{N}}^n$ containing $\{0, \ldots, 0\}$ we set $\mathcal{W} = ((\mathcal{D}_1, \ldots, \mathcal{D}_n), \mathfrak{W})$ and we define the set of *p-adic $\mathcal{W}$-points* as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}_p) = \{P \in X \mid \mathrm{mult}_p(P) \in W\},$$

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and the set of *$\mathcal{W}$-points over* $\mathbb{Z}$ as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{P \in X \mid \mathrm{mult}_p(P) \in W \text{ for all primes } p\}.$$

Multiplicities on toric varieties

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We take X to be a compact smooth split toric variety.We let $D_1, \ldots, D_n$ be the torus-invariant prime divisors $D_i = \{x_i = 0\}$ (on $\mathbb{P}^{n-1}$: coordinate hyperplanes).

Multiplicities on toric varieties

We can represent a point on a toric variety $X(\mathbb{Q})$ by its Cox coordinates $P = (a_1 : \cdots : a_n)$, corresponding to the D_i. By taking the coordinates in $\mathbb{Z}$ in primitive form (for $\mathbb{P}^{n-1}$ this is just $\gcd(a_1, \ldots, a_n) = 1$) we have we have $a_i \in \mathbb{Z}$ and

$$\mathrm{mult}_p(P) = (v_p(a_1), \ldots, v_p(a_n)).$$

Examples of $\mathcal{W}$-points

- $\mathfrak{W} = \{0\}^k \times \overline{\mathbb{N}}^{n-k}$ gives the integral points with respect to $D_1, \ldots, D_k$:

$$\begin{aligned}(\mathcal{X}, \mathcal{W})(\mathbb{Z}) &= (\mathcal{X} \setminus \cup_{i=1}^k \mathcal{D}_i)(\mathbb{Z}) \\ &= \{(\pm 1 : \cdots : \pm 1 : a_{k+1} : \cdots : a_n)\}.\end{aligned}$$

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Let $m_1, \ldots, m_n \in \mathbb{N} - \{0\}$.

- $\mathfrak{W} = \{(w_1, \ldots, w_n) \colon m_i | w_i\}$ gives the **Darmon points**
$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(\pm a_1^{m_1} : \cdots : \pm a_n^{m_n})\}.$$

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- $\mathfrak{W} = \{(w_1, \ldots, w_n) \colon w_i = 0 \text{ or } w_i \geq m_i\}$ gives the **Campana points**

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon a_i \ m_i\text{-full}\}.$$

W-approximation

We say that $\mathcal{X}$ satisfies *(integral) $\mathcal{W}$-approximation* if the embedding

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) \hookrightarrow \prod_{p \text{ prime}} (\mathcal{X}, \mathcal{W})(\mathbb{Z}_p) \times X(\mathbb{R})$$

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$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) \hookrightarrow \prod_{p \text{ prime}} (\mathcal{X}, \mathcal{W})(\mathbb{Z}_p)$$

has dense image. This generalizes strong approximation, which is when $(\mathcal{X}, \mathcal{W})(\mathbb{Z})$ are the integral points.

When is this satisfied? Consider the fan of X in $\mathbb{Z}^d$ ($d = \dim X$). Then we get a homomorphism

$$\phi \colon \mathbb{N}^n \to \mathbb{Z}^d$$

sending $e_i \mapsto u_i$, where u_i is the ray generator associated to D_i.

W-approximation for toric varieties

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$$N_W^+ \subset \mathbb{Z}^d$$

and a subgroup

$$N_W \subset \mathbb{Z}^d.$$

W-approximation for toric varieties

Theorem (B.M.,2023)

1. $\mathcal{X}$ satisfies $\mathcal{W}$-approximation off ∞ if and only if $N_W = \mathbb{Z}^d$,
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As N_W and N_W^+ are easy to compute, it is easy to decide whether $\mathcal{W}$-approximation holds.

Implications of the theorem

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Corollary

Strong approximation holds off ∞ with respect to $D_1, \ldots, D_k$ if and only if $X \setminus \cup_{i=1}^k D_i$ is simply connected as a complex manifold.

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Corollary

Strong approximation holds off ∞ with respect to $D_1, \ldots, D_k$ if and only if $X \setminus \cup_{i=1}^k D_i$ is simply connected as a complex manifold. This comes from the isomorphism

$$\mathbb{Z}^d / N_W \cong \pi_1(X \setminus \cup_{i=1}^k D_i).$$

Corollary (B.M.,2023)

$\mathcal{W}$-approximation holds for Darmon points if and only if there are no (nontrivial) finite covers $Y \to X$ ramified only over the D_i with ramification multiplicity $e_i | m_i$ at the D_i.

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In particular: if $\gcd(m_i, m_j) = 1$ for all $i \neq j$ then $\mathcal{W}$-approximation holds for Darmon points, and if $X = \mathbb{P}^n$ then the converse also holds.

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Example: if $X = \mathbb{P}^1$ and $m_1, m_2 = 2$, then $\mathcal{X}$ does not satisfy $\mathcal{W}$-approximation, as 2 mod 5 is not of the form $\pm a^2$ mod 5, but $(2:1) \in (\mathcal{X}, \mathcal{W})(\mathbb{Z}_5)$ as $2 \in \mathbb{Z}_5^\times$.

Proof sketch for projective space

If $p_1, \ldots, p_r$ are primes and $P_i = (a_{p_i,1}, \ldots, a_{p_i,n}) \in (\mathcal{X}, \mathcal{W})(\mathbb{Z}_{p_i})$, we want to find $Q \in (\mathcal{X}, \mathcal{W})(\mathbb{Z})$ approximating each to order p_i^N.

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$$Q' = \prod_{i=1}^{r} (p^{v_p(a_{p_i,1})}, \ldots, p^{v_p(a_{p_i,n})}).$$

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$$Q' = \prod_{i=1}^{r} (p^{v_p(a_{p_i,1})}, \dots, p^{v_p(a_{p_i,n})}).$$

Then Q' has the right multiplicities for all primes p_i and has multiplicity 0 for all other primes. This shows we can assume all multiplicities are 0.

Proof sketch for projective space

Let $w_1, \dots w_k \in \mathfrak{W}$ generate $\mathbb{Z}^d$. Then the linear map $\mathbb{Z}^k \to \mathbb{Z}^d$ is surjective, and thus the associated map

$$(\mathbb{Z}/p_i^N\mathbb{Z})^k \to (\mathbb{Z}/p_i^N\mathbb{Z})^d$$

is as well. So at each prime p_i we can find a point $Q_i \in \mathcal{X}(\mathbb{Z})$ which satisfies the $\mathcal{W}$-condition at p_i.

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is as well. So at each prime p_i we can find a point $Q_i \in \mathcal{X}(\mathbb{Z})$ which satisfies the $\mathcal{W}$-condition at p_i.The only obstacle now is to lift these points modulo p_i^N to a $\mathcal{W}$-point Q over $\mathbb{Z}$. For all i Dirichlets theorem on arithmetic progressions gives infinitely many primes q_i which are 1 mod p_j^N and have any given residue class q_i mod p_i. We use this to construct Q.

In general

The results transfer verbatim to number fields, and after slight modification also for function fields of curves.