Counting special points on toric varieties

Boaz Moerman

Utrecht University

Introduction

In number theory and arithmetic geometry, one of the main topics is the study of homogeneous Diophantine equations. These are systems of equations

$$f_1(X_1,\ldots,X_n)=\cdots=f_k(X_1,\ldots,X_n)=0,$$

where the functions $f_1,\ldots,f_k$ are homogeneous polynomials with integer coefficients in variables $X_1,\ldots,X_n$. For example,

$$X^2+Y^2-Z^2=0$$

is the equation describing a cone in $\mathbb{R}^3$. A central goal is to describe the integer solutions of such equations. In particular, how many such solutions exist and how common are they? In the example of the circle given, there are infinitely many solutions $(X,Y,Z)=(a,b,c)\in\mathbb{Z}^3$ given by Pythagorean triples. A classical result by Lehmer [1] states that the number of such solutions with $\gcd(a,b,c)=1$ and $\max(|a|,|b|,|c|)\leq B$ is asymptotic to $\frac{4}{\pi}B$.
By viewing such solutions as rational points on the projective variety determined by the equations, we can use tools from algebraic geometry to determine how common solutions to Diophantine equations are. In particular, if the variety determined by the Diophantine equation is Fano, then Manin's conjecture [2] gives a precise prediction for the number of solutions of bounded height. We extend this conjecture to a broad class of special solutions of Diophantine equations, such as squarefree solutions, coprime solutions, and squareful solutions. We then show that this extension is true for split toric varieties, such as projective space.

Projective space and toric varieties

In geometry, projective space is a foundational object of study. The rational points on the projective space $\mathbb{P}^{n-1}$ can be described using homogeneous coordinates: they are the points

$$(a_1:\cdots:a_n),$$

such that $a_1,\ldots,a_n\in\mathbb{Z}$ and $\gcd(a_1,\ldots,a_n)=1$. A natural extension of projective space is given by the class of smooth (split) toric varieties. These include products of projective spaces such as the quadric

$$\mathbb{P}^1\times\mathbb{P}^1\cong\{XY-ZW=0\}\subset\mathbb{P}^3,$$

as well as projective space blown up at a point. Toric varieties have Cox coordinates, which generalizes homogeneous coordinates. Using these coordinates, rational points on split toric varieties have a similar description, but with a different gcd condition. For example, on $\mathbb{P}^1\times\mathbb{P}^1$, the Cox coordinates for a rational point are of the form $(a_1:a_2:a_3:a_4)$, for integers a_1,a_2,a_3,a_4 satisfying $\gcd(a_1,a_2)=\gcd(a_3,a_4)=1$.

M-points on toric varieties

Now we introduce special subsets of the rational points on the toric variety X. For a rational point $Q=(a_1:\cdots:a_n)$ (with nonzero coordinates) and a prime p, we can take out the factors of p to get

$$(a_1:\cdots:a_n)=(p^{m_1}u_1:\cdots:p^{m_n}u_n),$$

where p does not divide $u_1,\ldots,u_n$. We define the **multiplicity** of Q at p to be

$$\operatorname{mult}_p(Q)=(m_1,\ldots,m_n).$$

Let $M\subset\mathbb{N}^n$ be a subset containing $(0,\ldots,0)$. Then a rational point $Q\in X(\mathbb{Q})$ is an **M-point** if $\operatorname{mult}_p(Q)\in M$ for all prime numbers p. We denote the set of M-points on X by $(X,M)(\mathbb{Z})$.

Examples of M-points

By varying M, we can describe several arithmetically interesting sets of points. For instance:

- $M=\mathbb{N}^t\times\{0\}^{n-t}$ gives
  $(X,M)(\mathbb{Z})=\{(a_1:\cdots:a_t:\pm1:\cdots:\pm1)\colon a_i\in\mathbb{Z}\setminus\{0\}\}$.
- $M=\{0,1\}^n$ gives
  $(X,M)(\mathbb{Z})=\{(a_1:\cdots:a_n)\colon a_i \text{ is squarefree}\}$.

If we fix a tuple of positive integers $\mathbf{m}=(m_1,\ldots,m_n)$, then

- If $M=\{(w_1,\ldots,w_n)\colon m_i|w_i\}$, then
  $$(X,M)(\mathbb{Z})=\{(\pm a_1^{m_0}:\cdots:\pm a_n^{m_n})\colon a_i\in\mathbb{Z}\setminus\{0\}\}$$
  are the **Darmon points** for $\mathbf{m}$.
- If $M=\{(w_1,\ldots,w_n)\colon w_i=0 \text{ or } w_i\geq m_i\}$, then
  $$(X,M)(\mathbb{Z})=\{(a_1:\cdots:a_n)\colon a_i \text{ is } m_i\text{-full}\}$$
  are the **Campana points** for $\mathbf{m}$. An integer a is said to be m-full if for every prime number dividing a, p^m also divides a.
- If $M=\{(w_1,\ldots,w_n)\colon\sum_{i=1}^n\frac{w_i}{m_i}\geq1\}\cup\{(0:\cdots:0)\}$, then $(X,M)(\mathbb{Z})$ are the **weak Campana points** of $\mathbf{m}$. Of particular note is when $m_1=\cdots=m_n=m$ for some integer m. In this case
  $$(X,M)(\mathbb{Z})=\left\{(a_1:\cdots:a_n)\colon\prod_{i=1}^n a_i\neq0 \text{ is } m\text{-full}\right\}.$$

Heights on varieties

We want to be able to quantify how common M-points are. For this we use the height. Let $Q=(a_1:\cdots:a_n)$ be a point on projective space. The **height** of Q is defined as

$$H(Q)=\max(|a_1|,\ldots,|a_n|).$$

If X is a variety with a given embedding into projective space, then we can restrict the height to X to get a height

$$H\colon X(\mathbb{Q})\to\mathbb{R}_{>0}.$$

M-points of bounded height

For a smooth split toric variety X with a given embedding into projective space, and a set $M\subset\mathbb{N}^n$, we define the counting function

$$N_{(X,M)}(B)\coloneqq\{Q\in(X,M)(\mathbb{Z})\mid H(Q)\leq B\}$$

for all real numbers B. Our main result is the following.

Theorem (M. 2025)[3]

Let X and M be as above, and assume that there exist positive integers $d_1,\ldots,d_n$ such that $(d_1,0,\ldots,0),(0,d_2,0,\ldots,0),\ldots,(0,\ldots,0,d_n)\in M$. Then there exist positive real numbers a,b,θ such that

$$N_{(X,M)}(B)=B^a\left(Q(\log B)+O\left(B^{-\theta}\right)\right)$$

as $B\to\infty$, where Q is a polynomial of degree $b-1$. The numbers a and b are explicit with a geometric interpretation. Furthermore, under mild hypotheses, the leading coefficient of Q is explicitly computed.

This generalizes results on Campana points by Pieropan and Schindler [4], which in turn generalizes results by Salberger on rational points [5]. These results are the first results on M-points of bounded height outside Campana and Darmon points.

The constants a and b

Let $\Gamma_M\subset\mathbb{N}^n$ be the finite set of minimal elements of M (with respect to the partial order on $\mathbb{N}^n$), and define the group of (torus-invariant) divisors on X and (X,M):

$$\operatorname{Div}_T(X)=\bigoplus_{i=1}^n\mathbb{Z}[D_i],\quad \operatorname{Div}_T(X,M)=\bigoplus_{\mathbf{m}\in\Gamma_M}\mathbb{Z}[\tilde{D}_{\mathbf{m}}],$$

where $D_i\subset X$ is the zero locus of the i-th coordinate and $\tilde{D}_{\mathbf{m}}$ is just a symbol. Consider the pullback homomorphism

$$\operatorname{pr}^*\colon\operatorname{Div}_T(X)\to\operatorname{Div}_T(X,M)$$
$$D_i\mapsto\sum_{\mathbf{m}\in\Gamma_M}m_i\tilde{D}_{\mathbf{m}}.$$

Let $\operatorname{Pic}(X,M)=\operatorname{Div}(X,M)/\operatorname{pr}^*\{\text{principal divisors on } X\}$ be the Picard group of (X,M) and let $K_{(X,M)}=-\sum_{\mathbf{m}\in\Gamma_M}\tilde{D}_{\mathbf{m}}$ be the Canonical divisor of (X,M). On projective space, a divisor $\sum_{i=1}^n a_iD_i$ is principal exactly when $\sum_{i=1}^n a_i=0$. In the vector space $\operatorname{Pic}(X,M)\otimes\mathbb{R}$, the effective cone $\operatorname{Eff}(X,M)$ is the cone generated by the elements $\tilde{D}_{\mathbf{m}}$ for $\mathbf{m}\in\Gamma_M$. Using the effective cone and the canonical divisor, we can determine a and b. The embedding of X in projective space comes from a divisor $L\in\operatorname{Div}(X)$ (for projective space, this is simply D_i). Now

$$a=\min\{t\in\mathbb{R}\mid t\operatorname{pr}^*L+K_{(X,M)}\in\operatorname{Eff}(X,M)\}$$

and b is the codimension of the minimal face of $\operatorname{Eff}(X,M)$ containing $a\operatorname{pr}^*L+K_{(X,M)}$.

Explicit examples

In many cases, the constants a,b are easy to compute.

Corollary

Let $X=\mathbb{P}^{n-1}$ and assume $(m_1,0,\ldots,0),\ldots,(0,\ldots,0,m_n)\in M$ for positive integers $m_1,\ldots,m_n$ and suppose $\sum_{i=1}^n\frac{w_i}{m_i}\geq1$ for all nonzero $\mathbf{w}\in M$. Then

$$a=\sum_{i=1}^n\frac{1}{m_i},\quad b=\#\left\{(w_1,\ldots,w_n)\in M\mid\sum_{i=1}^n\frac{w_i}{m_i}=1\right\}-n+1.$$

The hypothesis in the corollary is satisfied for Darmon points and (weak) Campana points. In particular,

- For Darmon and Campana points we have $b=1$, so there is no logarithmic factor appearing for such points on projective space.
- For weak Campana points with $m_1=\cdots=m_n=m$, we have
  $$b=\binom{m+n-1}{n-1}-\binom{m-1}{n-1}-n+1.$$

As an example, if we consider weak Campana points on the projective plane $\mathbb{P}^2$ with $m_1=m_2=m_3=2$, the theorem implies

$$\{(a:b:c)\in\mathbb{P}^2(\mathbb{Q})\mid abc\neq0 \text{ is squareful, } H(a:b:c)\leq B\}$$
$$=B^{3/2}(Q(\log B)+O(B^{-\theta})),$$

where Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}}\left(1-p^{-1}\right)^6\left(\frac{1-p^{-3/2}}{\left(1-p^{-1/2}\right)^3}-3p^{-1/2}\right)\approx0.862.$$

M-points on other varieties

While we focused on toric varieties over the rational numbers in this poster, it is possible to study M-points on other varieties and over number fields. For pairs (X,M) which are "rationally connected", I have formulated a conjecture for the number of M-points of bounded height, which generalizes Manin's conjecture as well as the theorem presented here.

References

[1] D. N. Lehmer. Asymptotic evaluation of certain totient sums. *Am. J. Math.*, 22:293–335, 1900.

[2] Victor V. Batyrev and Yuri I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.

[3] Boaz Moerman. Generalized campana points of bounded height, 2025. to appear.

[4] Marta Pieropan and Damaris Schindler. Hyperbola method on toric varieties. *J. Éc. polytech. Math.*, 11:107–157, 2024.

[5] Per Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque*, (251):91–258, 1998.