M-points and adelic approximation

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Origin

The study of weak and strong approximation has a long and rich history, dating back to the Chinese remainder theorem from the 5th century. Stated in modern language, the Chinese remainder theorem states that the diagonal embedding

$$\mathbb{Z} \hookrightarrow \prod_{p \text{ prime}} \mathbb{Z}_p$$

has a dense image. An algebraic geometric counterpart to this is the fact that

$$\mathbb{C}[t] \hookrightarrow \prod_{a \in \mathbb{C}} \mathbb{C}[[t-a]]$$

has dense image. This means that through any set of points on the plane, with different x-coordinates, there is a polynomial through these points. Furthermore, the first n derivatives at those points can be freely specified. The general theory considers more general fields than $\mathbb{Q}$ and $\mathbb{C}(t)$ and considers other different varieties than $\mathbb{A}^1$.

Summary

We extend this by putting intersection conditions on the points considered, generalizing the notions of integral points and Campana points. This notion allows for the study of intermediate forms between weak and strong approximation. Furthermore, it gives insight into special sets of rational points, such as squarefree points or squareful points. We mainly study this notion on split toric varieties.

$\mathcal{M}$-points

Let K be either a number field and $C = \operatorname{Spec} \mathcal{O}_K$, or let $K = k(C)$ for any field k and a projective regular curve C over k. Let X be a proper variety over K with a proper model $\mathcal{X}$ over C. Let $D_1, \dots, D_n$ be prime divisors on X with Zariski closures $\mathcal{D}_1, \dots, \mathcal{D}_n$ in $\mathcal{X}$ and let $\mathfrak{M} \subset \overline{\mathbb{N}}^n$ be a subset containing $(0, \dots, 0)$, where $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Consider the pair $(\mathcal{X}, \mathcal{M})$ where $\mathcal{M} = ((\mathcal{D}_1, \dots, \mathcal{D}_n), \mathfrak{M})$. For a point $v \in B$, let $\mathcal{O}_v$ be the completion of $\mathcal{O}_C$ at v. We choose a uniformizer $\pi_v \in \mathcal{O}_v$. A point $\mathcal{P} \in \mathcal{X}(\mathcal{O}_v)$ is a **v-adic $\mathcal{M}$-point on** $(\mathcal{X}, \mathcal{M})$ if $\operatorname{mult}_v(\mathcal{P}) := (n_v(\mathcal{D}_1, \mathcal{P}), \dots, n_v(\mathcal{D}_n, \mathcal{P}))$ is contained in $\mathfrak{M}$. Here $n_v(\mathcal{D}, \mathcal{P})$ is the intersection multiplicity of $\mathcal{D}$ and $\mathcal{P}$ at v. This is defined as the integer n such that (π_v^n) is the ideal in $\mathcal{O}_v$ corresponding to the effective divisor $\mathcal{P}^*\mathcal{D}$. Let $B \subset C$ be an open subset. A point $\mathcal{P} \in X(K) = \mathcal{X}(B)$ is an **$\mathcal{M}$-point on** $(\mathcal{X}, \mathcal{M})$ **over** B if it is a v-adic $\mathcal{M}$-point for all $v \in B$. We denote the sets of $\mathcal{M}$-points by $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ and $(\mathcal{X}, \mathcal{M})(B)$, respectively.

$\mathcal{M}$-points on toric pairs

Let X be a smooth complete split toric variety over K, let $\mathcal{X}$ be the model over B obtained from the fan of X and let $D_1, \dots, D_n$ be the torus invariant prime divisors. For example, we can take $X = \mathbb{P}_K^{n-1}$ and $\mathcal{X} = \mathbb{P}_B^{n-1}$ with $D_1, \dots, D_n$ the coordinate hyperplanes. For any $\mathfrak{M} \subset \overline{\mathbb{N}}^n$, we call $(\mathcal{X}, \mathcal{M})$ a **toric pair**. For $v \in B$, a point $\mathcal{P} \in \mathcal{X}(\mathcal{O}_v)$ can be written in Cox coordinates as

$$\mathcal{P} = (u_1 \pi_v^{m_1} : \cdots : u_1 \pi_v^{m_n})$$

for units $u_1, \dots, u_n \in \mathcal{O}_v^\times$ and $m_1, \dots, m_n \in \overline{\mathbb{N}}$ which vanish outside of some cone. (For projective space, these are just homogeneous coordinates such that $\min(m_1, \dots, m_n) = 0$.) In this notation the multiplicity is simply

$$\operatorname{mult}_v(\mathcal{P}) = (m_1, \dots, m_n).$$

Examples of $\mathcal{M}$-points

In particular if $B = \operatorname{Spec} R$, where R is a PID, then some examples of sets of $\mathcal{M}$-points are

- $\mathfrak{M} = \overline{\mathbb{N}}^t \times \{0\}^{n-t}$ gives $(\mathcal{X}, \mathcal{M})(R) = \{(a_1 : \cdots : a_t : u_{t+1} : \cdots : u_n) : a_i \in R, u_i \in R^\times\}$.
- $\mathfrak{M} = \{0, 1\}^n$ gives $(\mathcal{X}, \mathcal{M})(R) = \{(a_1 : \cdots : a_n) : a_i \text{ squarefree}\}$.

If we fix positive integers $m_1, \dots, m_n$, then

- If $\mathfrak{M} = \{(w_1, \dots, w_n) : m_i | w_i\}$, then
$$(\mathcal{X}, \mathcal{M})(R) = \{(u_1 a_1^{m_0} : \cdots : \pm u_n a_n^{m_n}) : a_i \in R, u_i \in R^\times\}$$
are the **Darmon points**.
- If $\mathfrak{M} = \{(w_1, \dots, w_n) : w_i = 0 \text{ or } w_i \geq m_i\}$, then
$$(\mathcal{X}, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \cdots : a_n) : a_i \ m_i\text{-full}\}$$
are the **Campana points**. An element $a \in R$ is said to be m-full if for every prime ideal $\mathfrak{p}$ with $a \in \mathfrak{p}$, we also have $a \in \mathfrak{p}^m$.

Integral $\mathcal{M}$-approximation

Recall that the places of K is just C if K is a function field and for a number field is C together with the embeddings $K \to \mathbb{R}$ and $K \to \mathbb{C}$ (up to conjugacy). We denote the set of places by Ω_K. We say that $(\mathcal{X}, \mathcal{M})$ **satisfies integral* $\mathcal{M}$-approximation off** T if the diagonal embedding

$$(\mathcal{X}, \mathcal{M})(C \setminus T) \to \prod_{v \in \Omega_K \setminus T} (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$$

has dense image. Here we define $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) = X(K_v)$ when v is an infinite place.

*The more fundamental notion is M-approximation, which is independent of the model and is better behaved, but harder to define.

Monoids coming from a toric pair

Let X be a split toric variety over K with a fan in $N \cong \mathbb{Z}^d$ and ray generators $n_{\rho_1}, \dots, n_{\rho_n}$. For example, if $X = \mathbb{P}^{n-1}$, then $d = n-1$ and $n_{\rho_i} = e_i$ if $i \neq n$ and $n_{\rho_n} = -e_1 - \cdots - e_{n-1}$. Consider the homomorphism of monoids $\phi : \mathbb{N}^n \to N$ sending e_i to n_{ρ_i}. For a toric pair $(\mathcal{X}, \mathcal{M})$, let $N_{\mathcal{M}}^+$ and $N_{\mathcal{M}}$ be the monoid and the lattice in $\mathbb{Z}^d$ generated by $\phi(\mathfrak{M})$. The quotient $N/N_{\mathcal{M}}$ can be thought of as a 'fundamental group' of $(\mathcal{X}, \mathcal{M})$.

Furthermore, let $\rho(K) = \{1\}$ if K is a number field. If K is a function field and k is the field of constants of C, let $\rho(K) \subset \mathbb{N}^*$ be the monoid generated by the primes p such that

1. p is different from the characteristic and
2. for every finite extension l/k, p does not divide $|l/k|$.

We now have a characterisation of integral $\mathcal{M}$-approximation on toric pairs, extending the work of Nakahara and Streeter for Campana points on projective space over a number field [1].

$\mathcal{M}$-approximation for toric pairs

Let T be any nonempty set of places. A toric pair $(\mathcal{X}, \mathcal{M})$ satisfies

1. integral $\mathcal{M}$-approximation off T if $N_{\mathcal{M}}$ has finite index in N and $|N/N_{\mathcal{M}}| \in \rho(K)$. If $\operatorname{Pic}(C)$ is finitely generated, then the converse is also true.
2. The pair $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation if and only if $N_{\mathcal{M}}^+ = N$.

For simplicity assume that $\operatorname{Pic}(C)$ is finitely generated.

Consequences

1. An affine toric variety U over K satisfies strong approximation if and only if it does not have nonconstant global sections and $\operatorname{Pic}(U)$ is torsion-free. If K has characteristic zero, U satisfies strong approximation if and only if U is simply connected.
2. If $(\mathcal{X}, \mathcal{M})$ is a toric pair associated to Campana points or squarefree points, then integral $\mathcal{M}$-approximation is always satisfied.
3. If $(\mathcal{X}, \mathcal{M})$ is a toric pair associated to Darmon points and K has characteristic 0, then integral $\mathcal{M}$-approximation is satisfied if and only if the associated root stack is simply connected. For $\mathbb{P}^{n-1}$, this is equivalent to $\gcd(m_i, m_j) = 1$ for all distinct i, j. For other toric varieties this is a sufficient criterion for $\mathcal{M}$-approximation.

Thin sets

If $(\mathcal{X}, \mathcal{M})$ satisfies $\mathcal{M}$-approximation, then $(\mathcal{X}, \mathcal{M})(C)$ is dense in X in a very strong sense. This raises the question whether this set is also not thin. For a variety X over K, a set $A \subset X(K)$ is **thin** if it is contained in a finite union of sets of the form

1. $Z(K)$, for a closed subvariety Z in X and
2. $\pi(Y(K))$, for a generically finite morphism $Y \to X$ of degree at least 2.

We proved a theorem relating $\mathcal{M}$-approximation and thin sets, extending results by Nakahara and Streeter about Campana points over number fields [1]. Their result in turn generalizes a theorem of Colliot-Thélène and Ekedahl from the late 1980s [2].

$\mathcal{M}$-approximation and a $\mathcal{M}$-Hilbert property

Let K be a global field and let $(\mathcal{X}, \mathcal{M})$ be a pair satisfying integral $\mathcal{M}$-approximation off a finite set of places T. Then $(\mathcal{X}, \mathcal{M})(C)$ is not thin.

Furthermore, if $(\mathcal{X}, \mathcal{M})$ is a toric pair, then $(\mathcal{X}, \mathcal{M})(C)$ is not thin if and only if $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation.

$\mathcal{M}$-Hilbert property over other fields

The above result does not extend to fields with $\rho(K) \neq \{1\}$. For such fields any toric pair $(\mathcal{X}, \mathcal{M})$ with $1 \neq |N/N_{\mathcal{M}}| \in \rho(K)$ the set $(\mathcal{X}, \mathcal{M})(C)$ is thin even though the pair satisfies integral $\mathcal{M}$-approximation off any nonempty finite set of places T. Does the result still hold if $\rho(K) = \{1\}$ or if $T = \varnothing$?

References

[1] Masahiro Nakahara and Sam Streeter. Weak approximation and the Hilbert property for Campana points, 2023.

[2] Jean-Pierre Serre. *Topics in Galois Theory.* Research Notes in Mathematics. CRC Press, 2016.