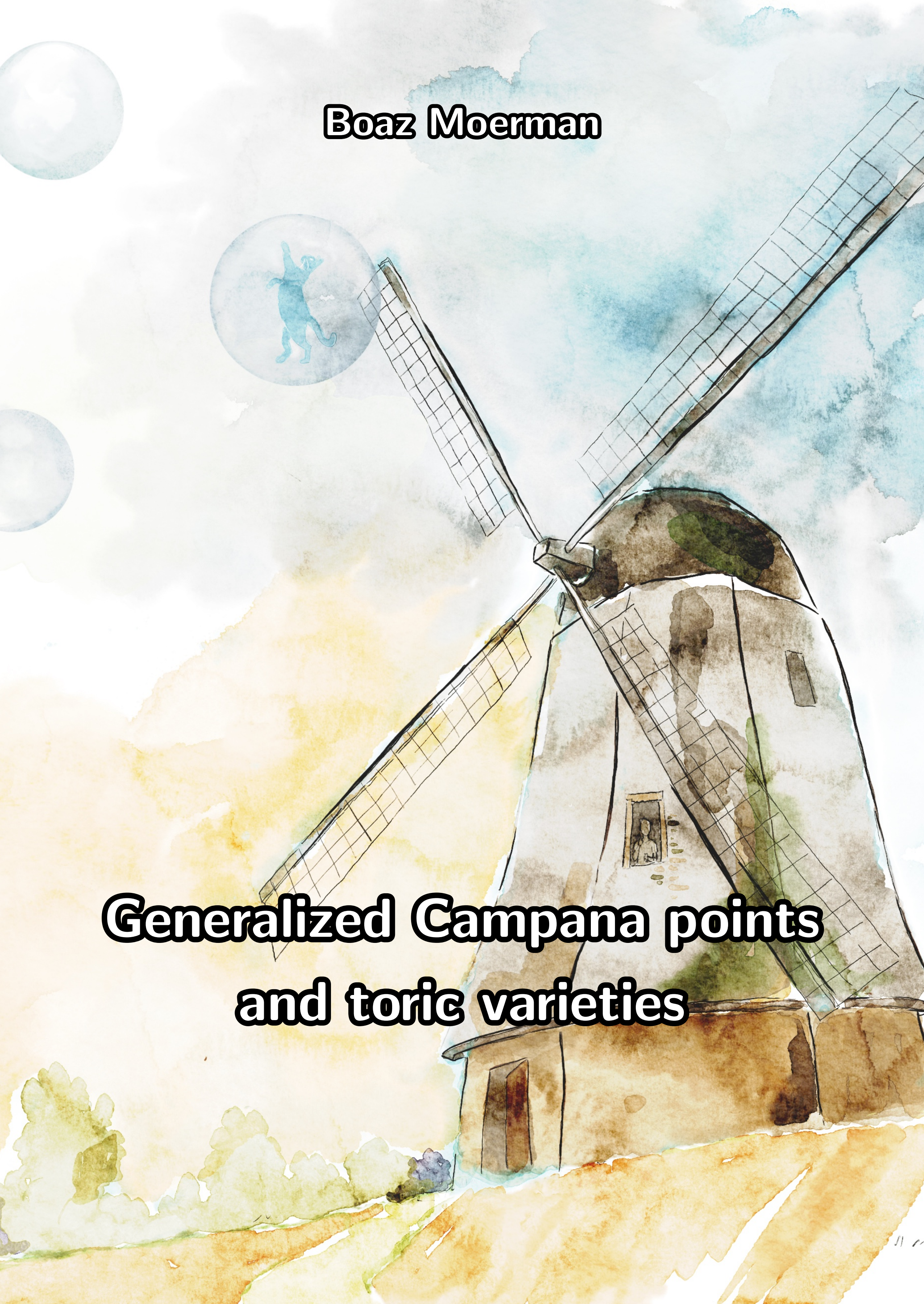

Boaz Moerman

Generalized Campana points and toric varieties

Generalized Campana points and toric varieties

Cover: Sara Mehidi

DOI: https://doi.org/10.33540/3072

ISBN: 978-90-393-7936-3

Generalized Campana points and toric varieties

Gegeneraliseerde Campanapunten en torische variëteiten
(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. ir. W. Hazeleger, ingevolge het besluit van het College voor Promoties in het openbaar te verdedigen op

dinsdag 16 september 2025 des middags te 2.15 uur

door
Boaz Moerman
geboren op 21 april 1998
te Nieuwegein

Promotor:
Prof. dr. C. F. Faber

Copromotor:
Dr. M. Pieropan

Beoordelingscommissie:
Dr. F. Balestrieri
Prof. dr. G. L. M. Cornelissen
Prof. dr. E. Peyre
Prof. dr. P. Salberger
Prof. dr. D. Schindler

This thesis has been financially supported by the Mathematical Institute of Utrecht University.

Contents

1. Introduction

In number theory, one of the main topics is the study of Diophantine equations: systems of polynomial equations with integer coefficients. A central goal is to describe the integer and rational solutions of such equations. This leads to several basic questions, such as:

1. Are there any solutions?
2. Is the set of solutions infinite?
3. When can solutions of the equations modulo a prime number p be lifted to integer solutions?

Over the decades, algebraic geometry has shown to be a very powerful tool in studying such questions. The system of equations defines an algebraic variety, and the integer solutions correspond to integral or rational points on the variety. By studying the geometry of this variety, much information is obtained on the solutions, and this has lead to the field of Diophantine geometry and the broader field of arithmetic geometry. This philosophy was captured beautifully by Hindry and Silverman in the introduction of [HS00] by the slogan

Geometry Determines Arithmetic

This approach to Diophantine equations has proven to be very successful, and has led to important results such as the Mordell-Weil Theorem [Wei28], the proof of the Mordell Conjecture by Faltings [Fal83] and the Modularity Theorem [BCD$^+$01].

In number theory, there is also a significant interest in special solutions of Diophantine equations, such as squarefree solutions, coprime solutions, and squareful solutions. The study of squarefree values of polynomials is an active research area, see for example [Hoo67; Fil92; Poo03; SW23]. Squareful numbers have also been extensively studied, such as in [ES34; Hea90; ZW12; Van12]. In this thesis, we set up a general geometric framework for studying special solutions of Diophantine equations, which we call $\mathcal{M}$-points. This allows us to apply the methods and language of arithmetic geometry to understand such solutions.

1.0.1 Campana points and related notions

The framework of $\mathcal{M}$-points subsumes the theory of Campana points. The theory of Campana points allows for a geometric study of squareful, and more generally m-full, solutions of Diophantine equations. Here we recall that an integer n is m-full if for every prime number p dividing n, p^m also divides n. Campana points can be viewed as integral points on a variety with respect to a weighted boundary divisor.

In recent years, Campana points have attracted a significant amount of attention, as many results and conjectures concerning rational points extend to Campana points. One such example is the Mordell Conjecture [Fal83], whose analogue for Campana points has been introduced by Campana [Cam05] and which was proven over function fields of characteristic 0 by him [Cam05] and proven recently in [KPS22] in positive characteristic. Over number fields the conjecture is implied by the *abc* Conjecture, see [Sme17, Appendix] for example. Recently in [BJ24] it was shown that the Kobayashi–Ochiai Theorem generalizes to Campana pairs, giving a higher dimensional analog of the Mordell Conjecture for the theory of Campana points over function fields. This result has found uses in proving scarcity of rational points on some surfaces [Sme17; KPS22] and certain threefolds [BJR24]. Campana points have also been used to show that the weakly special conjecture of Harris and Tschinkel [HT00] contradicts the *abc*-Conjecture [BCJ$^+$24].

Similarly, recently a generalization of Manin's conjecture on rational points of bounded height to the setting of Campana points has been introduced in [PSTVA21]. This conjecture is known for several varieties with appropriately chosen boundary divisor, such as diagonal hypersurfaces [Van12; BY21; Shu21; Shu22; BBK$^+$24], vector group compactifications [PSTVA21], norm forms [Str22], biequivariant compactifications of the Heisenberg group [Xia22] and wonderful compactifications [CLT$^+$25]. Furthermore, for the log-anticanonical height, this conjecture has been proven for complete toric varieties [PS24a; SS24] and certain complete intersections therein [PS24b]. In [Fai23; Fai25], a motivic analogue of Manin's conjecture for Campana points is proven for vector group compactifications and toric varieties.

Several other related types of rational points have been studied as well, such as Darmon points. For a Diophantine equation, the solutions of the equation by powers of integers (up to units) is a set of Darmon points. This terminology was coined in [MNS24], honoring Darmon's study of *M-curves* [Dar97]. In that paper, Darmon unconditionally proves the analog of the Mordell Conjecture for Darmon points over number fields and uses it to prove that generalized Fermat equations have finitely many solutions. Darmon points also appear in Campana's work as *morphismes orbifoldes divisibles* [Cam11a, Définition 2.4]. Darmon points are intrinsically connected to orbifolds through the root stack construction given in [Cad07], as we will explain in Section 3.5.

In other works [AV18; Str22] another variant of Campana points called weak Campana points is used in the study of analogues of the conjectures of Vojta [Voj87] and Manin [Pey95] on rational points.

1.0.2 $\mathcal{M}$-points

The notion of $\mathcal{M}$-points vastly generalizes integral points, (weak) Campana points and Darmon points and provides a common framework for studying these notions. To define such points, one needs to fix an integral model of the variety. Here the parameter set $\mathcal{M}$ encodes the boundary components on the integral model and the admissible intersection multiplicities, and the letter $\mathcal{M}$ was chosen in reference to the latter. The precise definition is given in Section 2.1.1. For example, in $\mathbb{A}^n_{\mathbb{Z}}$, for suitable choices of the parameter set $\mathcal{M}$, $\mathcal{M}$-points can describe points with integer coordinates that are squarefree, all cubes, or all coprime. On the geometric side, $\mathcal{M}$-

points can describe tuples of polynomials that are squarefree, have no simple zeroes, or are coprime. Many more examples are given in Section 2.1.4.

The main goal of this thesis is to study the following question: “How are $\mathcal{M}$-points distributed on a variety?”

There are different ways in which this question can be understood. In this thesis, we study two precise questions regarding $\mathcal{M}$-points:

- When does the set of $\mathcal{M}$-points satisfy M-approximation, the natural analogue of strong approximation?
- What is the number of $\mathcal{M}$-points of bounded height?

We study the first question in Chapter 2 and in Chapter 3, and the second question in Chapter 4 and Chapter 5. In both cases, we first set up general definitions and general results for $\mathcal{M}$-points, and afterwards focus on $\mathcal{M}$-points on split toric varieties.

Chapter 4 and Chapter 5 can be read independently from most of Chapter 2 and Chapter 3. More specifically, Chapter 4 only depends on the definition of pairs and $\mathcal{M}$-points introduced in Section 2.1 as well as the examples given in Section 2.1.4, while Chapter 5 additionally requires Cox coordinates and toric pairs as introduced in Section 3.1.

1.1 M-approximation

1.1.1 Weak and strong approximation

In number theory and algebraic geometry, the study of weak and strong approximation on algebraic varieties is an enduring area of research. These two properties are intimately related to the following question: given a system of polynomial equations

$$\{f_1 = \cdots = f_n = 0\}$$

with integer coefficients, does every solution of these equations in $\mathbb{Z}/n\mathbb{Z}$ lift to a solution over $\mathbb{Z}$? Such a lift exists as long as the variety defined by these equations satisfies strong approximation off the infinite place. If the variety X satisfies weak approximation off the infinite place, then every solution in $\mathbb{Z}/n\mathbb{Z}$ can be lifted to a nonzero solution of

$$\{F_1 = \cdots = F_n = 0\},$$

over $\mathbb{Z}$, where F_i is the homogenisation of the polynomial f_i.

Both weak and strong approximation serve as a natural extension of the Chinese Remainder Theorem. Using the language of the p-adic numbers, the Chinese Remainder Theorem is equivalent to the statement that

$$\mathbb{Z} \to \prod_{p \text{ prime}} \mathbb{Z}_p$$

has dense image. Strong approximation and weak approximation can both be formulated in a similar way. Let K be a number field or a function field of a curve, and write Ω_K for the set of places of K. Then a variety X over K satisfies weak approximation if the natural embedding

$$X(K) \hookrightarrow \prod_{v \in \Omega_K} X(K_v)$$

has dense image, where K_v is the completion of K with respect to the place v. In other words, X satisfies weak approximation if, for every finite collection S of places of K with chosen points $P_v \in X(K_v)$ for all $v \in S$, there exists a rational point $P \in X(K)$ approximating all P_v arbitrarily well. The variety X satisfies strong approximation if the rational point P can be chosen to be integral with respect to all places outside of S. For a precise description of weak and strong approximation, see Section 2.2.2.

Strong approximation was first studied by Eichler in [Eic38], where he studied the property for certain algebraic groups over number fields. In the 60's and 70's, his results were extended to all semisimple simply-connected algebraic groups [Kne66; Pla69; Mar77; Pra77]. Much more recently, other types of varieties have been shown to satisfy strong approximation (with Brauer-Manin obstruction), such as certain quadrics [BC08], toric varieties [WX12; CX18b; Wei21; San23a; Che24], certain quadratic fibrations [Xu15], certain homogeneous spaces [BD13; CH16; Col18; Dem22], certain affine hypersurfaces [DW17], groupic varieties [Cao18; CX18a] and certain complete intersections [CZ18]. There is also an extensive literature on weak approximation (with Brauer-Manin obstruction) for the aforementioned types of varieties [CSS87; CS89; Har95; Ski97; CL14; Luc14], as well as for certain abelian varieties [Wan96], certain del Pezzo surfaces [Vár08] and Châtelet surfaces [NR24]. Rationally connected varieties over a function field of a complex curve have been shown to satisfy weak approximation away from places of bad reduction [HT06].

1.1.2 M-approximation

As with integrality, the $\mathcal{M}$-condition is a local condition, so local-global principles such as strong approximation have natural analogues for $\mathcal{M}$-points. In Section 2.2.2 of Chapter 2, we introduce M-approximation and integral $\mathcal{M}$-approximation, which generalizes and interpolates between weak approximation and (integral) strong approximation. Here M encodes the boundary components on the variety and the same intersection multiplicities as $\mathcal{M}$. The precise definition is given in Section 2.1.1. A variety X satisfies *M-approximation* if, roughly speaking, for every finite set of places $S \subset \Omega_K$ and points $P_v \in X(K_v)$ for $v \in S$, there exists $P \in X(K)$ approximating all P_v, such that for every place $v \in \Omega_K \setminus S$, P satisfies the $\mathcal{M}$-condition at the place v. This is formalized in terms of density of rational points in an appropriate adelic space, introduced in Definition 2.2.1. Similarly to strong approximation, this property is independent of the choice of the integral model.

Integral $\mathcal{M}$-approximation is a variant on this notion obtained by letting $P_v \in X(K_v)$ be a local $\mathcal{M}$-point and requiring $P \in X(K)$ to be an $\mathcal{M}$-point. Under very mild conditions, M-approximation implies integral $\mathcal{M}$-approximation, see Section 2.2.2 and Proposition 3.2.20.

Integral $\mathcal{M}$-approximation extends the notion of weak Campana approximation which was introduced by Nakahara and Streeter in [NS24] and further studied in [NS24; CLT24]. However, M-approximation behaves very differently from the notion of Campana strong approximation as given in their follow up paper with Mitankin [MNS24]. Their notion of Campana strong approximation interpolates between strong approximation and integral strong approximation, rather than weak approximation.

1.1.3 M-approximation and the $\mathcal{M}$-Hilbert property

If X is an integral variety over a field K, then a subset $A \subset X(K)$ is thin if it is contained in a finite union of proper closed subvarieties and images of the set of rational points under generically finite maps $Y \to X$ of degree greater than 1, where Y is an integral variety. In positive characteristic this is less strict than the usual definition as given in [BFP14; Lug22], see Remark 2.2.17.

The fields considered in this thesis are PF fields (K, C). Here K is either a number field or the function field of a regular projective curve C over some field k. If K is a number field then $C = \operatorname{Spec} \mathcal{O}_K$ where $\mathcal{O}_K$ is the ring of integers. Such fields have a good notion of places, as explained in Section 1.3.4, and therefore allow for the study of local-global principles, which has been done previously in [Yam96; Yam02]. The terminology 'PF fields' is explained by the fact that such fields satisfy a product formula, see Remark 1.3.4.

Let X be a proper variety over a PF field (K, C). Let $B \subset C$ be an open subscheme. As is the case for integral points, we consider integral models over B in order to define $\mathcal{M}$-points over B in $X(K)$. If M is a parameter set defined by boundary components D_α and a set of admissible multiplicities $\mathfrak{M}$ as in Definition 2.1.1, then we can consider an proper integral model $\mathcal{X}$ over B with parameter set $\mathcal{M}$ obtained by taking proper integral models $\mathcal{D}_\alpha \subset \mathcal{X}$ of the boundary components. We then call (X, M) a pair and $(\mathcal{X}, \mathcal{M})$ an integral model of (X, M) over B and write the set of $\mathcal{M}$-points as $(\mathcal{X}, \mathcal{M})(B)$. In the literature on Campana points, the boundary components D_α are divisors. We do not impose this restriction however, and allow the components D_α to be arbitrary closed subschemes.

The first theorem shows that integral $\mathcal{M}$-approximation implies that the set of $\mathcal{M}$-points is Zariski dense, unless this set is empty. If K is a global field it also shows that the set of $\mathcal{M}$-points is not thin.

Theorem 1.1.1. *Let (X, M) be a pair over a PF field (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over an open subscheme $B \subset C$. Assume that X is a geometrically reduced variety and that D_α does not contain an irreducible component of X for any $\alpha \in \mathcal{A}$.*

If $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off a finite set of places $T \subset \Omega_K$ and $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$, then X is geometrically integral and $(\mathcal{X}, \mathcal{M})(B)$ is Zariski dense. If furthermore K is a global field then $(\mathcal{X}, \mathcal{M})$ satisfies the $\mathcal{M}$-Hilbert property over B, i.e., $(\mathcal{X}, \mathcal{M})(B)$ is not thin in $X(K)$.

This theorem is a generalisation to $\mathcal{M}$-points of a result of Nakahara and Streeter [NS24, Theorem 1.1] which shows that weak weak Campana approximation implies the Campana Hilbert property. Their result is in turn a generalisation of a theorem of Colliot-Thélène and Ekedahl [Ser08, Theorem 3.5.7], which states that weak weak approximation implies the Hilbert property. Theorem 1.1.1 also generalizes all of these results from number fields to global fields, and removes the assumption that the variety is normal or even integral. Furthermore, the theorem even extends to function fields of curves over infinite fields, albeit with a weaker conclusion. The stronger conclusion obtained for global fields does not need to hold for such function fields, as shown in Corollary 3.3.11. This corollary shows that, for a function field K of a curve over an algebraically closed field, integral $\mathcal{M}$-approximation need not imply the $\mathcal{M}$-Hilbert property. In particular, this gives the first examples of varieties which satisfy strong approximation, but for which the set of integral points is both Zariski dense and thin.

If the field K is a number field and X is geometrically irreducible, then the proof of Theorem 1.1.1 closely follows the proof of Nakahara and Streeter. The main idea is to use the Lang–Weil bounds [LW54] to show that, for a generically finite morphism $Y \to X$ of degree greater than 1, the image of $Y(K_v)$ in $X(K_v)$ is too small to contain $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$. For global function fields, the proof is similar but more complicated, due to the existence of inseparable morphisms $Y \to X$. For these morphisms, we cannot apply the Lang–Weil bounds, and instead we prove and use Lemma 2.2.19, which implies that the image of $Y(K_v) \to X(K_v)$ is nowhere dense if X is smooth.

For function fields of curves over infinite fields, the main difficulty in proving Zariski density of $(\mathcal{X}, \mathcal{M})(B)$ is that we do not know whether $X(K_v)$ contains a smooth point for some place v. We show this by invoking recent results by Moret-Bailly on rational points in fibres [Mor20] and combining this with Hensel's Lemma.

Remark 1.1.2. The assumption that X is geometrically reduced is necessary for the conclusion of Theorem 1.1.1 to hold. Example 2.2.20 gives an example of an integral curve such that $X(K_v) = X(K)$ consists of a single point for every place v, so that X satisfies weak approximation but $X(K)$ is not Zariski dense in X.

1.1.4 M-approximation for split toric varieties

In Chapter 3, we focus on studying M-approximation for split toric varieties X. More specifically, we consider pairs (X, M) where the boundary components used in defining M are the torus-invariant prime divisors $D_1, \dots, D_n$. We will call such a pair (X, M) a *toric pair*. In this setting we give a necessary and sufficient criterion for M-approximation to hold off a given set of places T. To state the theorem, we will need the set

$$\rho(K, C) = \left\{ n \in \mathbb{N}^* \mid \mathcal{O}_v^\times \xrightarrow{(\cdot)^n} \mathcal{O}_v^\times \text{ is surjective for all } v \in \Omega_K \right\}.$$

An integer n lies in this set if for every $v \in \Omega_K$, every unit in $\mathcal{O}_v$ admits an n-th root. This set has not been studied before to the author's knowledge. This set is explicitly computed in Lemma 3.2.12. In particular, $\rho(K, C) = \{1\}$ if K is a global field, and $\rho(K, C) = \mathbb{N} \setminus \operatorname{char}(K)\mathbb{N}$ if K is the function field of a curve over a separably closed field.

Theorem 1.1.3. *Let (K, C) be a PF field and let (X, M) be a toric pair where X is a normal complete split toric variety over K with co-character lattice N. Let $T \subset \Omega_K$ be a nonempty finite set of places and let $N_M, N_M^+ \subset N$ be the lattice and the monoid as in Definition 3.2.2 and Definition 3.2.6. Then*

1. *(X, M) satisfies M-approximation off T if $|N : N_M| \in \rho(K, C)$. If $\operatorname{Pic}(C)$ is finitely generated, then the converse also holds.*

2. *Furthermore,* (X,M) *satisfies* M*-approximation if and only if* $N = N_M^+$.

This considerably generalizes Nakahara's and Streeter's result [NS24, Theorem 1.2(i)] from projective space to general split toric varieties, from Campana points to $\mathcal{M}$-points and from number fields to PF fields.

The proof of Nakahara and Streeter does not extend to the setting of $\mathcal{M}$-points, as their proof essentially uses the fact that for any two S-integers a, b such that b divides a, the integer $a^m b$ is m-full for any positive integer m. Since we consider sets $(\mathcal{X},\mathcal{M})(B)$ of points which can greatly differ from Campana points and can have much less structure, our proof of Theorem 1.1.3 takes a different approach. The proof is subdivided in two steps. First in Section 3.2.2 we prove results on the density of squarefree elements in rings of integers and on affine curves, which can be thought of as "squarefree strong approximation" on the affine line. Then we use Cox coordinates as introduced in [Cox95] to extend the results from the affine line to toric varieties.

Remark 1.1.4. The condition that $\mathrm{Pic}(C)$ is finitely generated is satisfied in many cases, such as when C is rational or when K is finitely generated over its prime field. The latter follows from Néron's generalisation of the Mordell-Weil Theorem [Con06, Corollary 7.2].

Remark 1.1.5. Note that M-approximation off a nonempty set of places only depends on the lattice N_M rather than on (X,M). This can be viewed as an analogue of purity of strong approximation as in [CLX19; CH20; Wei21; Che24]. In fact, this theorem shows that purity holds for strong approximation with respect to toric subvarieties.

By a classical theorem of Minchev [Min89, Theorem 1], of which we give a new proof in Corollary 2.2.21, a variety over a number field can only satisfy strong approximation off a finite set of places T if it is algebraically simply connected. The following consequence of Theorem 1.1.3 implies that, for split toric varieties, these two properties are actually equivalent.

Corollary 1.1.6. *Let* (K,C) *be a PF field of characteristic* 0*, let* $\overline{K}$ *be an algebraic closure of* K*, let* X *be a complete normal split toric variety over* K *and let* $V \subseteq X$ *be an open toric subvariety. Then:*

1. *For any nonempty finite set of places* T*,* V *satisfies strong approximation off* T *if* $\pi_1(V_{\overline{K}})$ *is finite and* $|\pi_1(V_{\overline{K}})| \in \rho(K,C)$*. If* $\mathrm{Pic}(C)$ *is finitely generated, then the converse also holds.*

2. *The variety* V *satisfies strong approximation if and only if* $V_{\overline{K}}$ *is simply connected and* $\mathcal{O}(V_{\overline{K}}) = \overline{K}$*.*

Remark 1.1.7. If $\rho(K,C) = 1$, then the first part of Corollary 1.1.6 states that for any nonempty set of places T, V satisfies strong approximation off T if and only if V is simply connected. On the other hand, if $\rho(K,C) = \mathbb{N}^*$ then V satisfies strong approximation off T if and only if its fundamental group is finite, or equivalently if and only if V does not have torus factors by [CLS11, Exercise 12.1.6].

In Section 3.4, we give two more characterisations of strong approximation on split toric varieties. Corollary 3.4.2 characterizes strong approximation in terms of the Picard group and is valid over any PF field. Corollary 3.4.4 implies that, over a number field, a smooth split toric variety satisfies strong approximation off a finite nonempty set of places if and only if the Brauer group modulo its constants vanishes. This strengthens the results of Cao and Xu in [CX18b] on strong approximation with Brauer-Manin obstruction for toric varieties over number fields when the toric variety is split toric. We strengthen their results by taking infinite places into consideration and we allow the ground field to be a function field. A similar result has recently also been shown over number fields by Santens in [San23a, Theorem 1.3], which implies that the algebraic Brauer-Manin obstruction is the only obstruction to strong approximation if every regular function on the variety $V_{\overline{K}}$ is constant.

By applying Theorem 1.1.3 to Campana points as defined in Definition 2.1.19, we obtain the following generalization of [NS24, Theorem 1.2(i)]:

Corollary 1.1.8. *Let (K, C) be a PF field, let X be a complete normal split toric variety and let $T \subset \Omega_K$ be a finite set of places. Let (X, M) be the toric pair corresponding to the Campana points on $(X, D_{\mathbf{m}})$ as defined in Definition 2.1.19, where $\mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{N}^* \cup \{\infty\})^n$ and*

$$D_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i.$$

Then (X, M) satisfies M-approximation off T if $X \setminus \lfloor D_{\mathbf{m}} \rfloor$ satisfies the conditions for strong approximation given in Corollary 1.1.6 or Corollary 3.4.2. If furthermore $\mathrm{Pic}(C)$ is finitely generated or $T = \emptyset$, then the converse also holds. In particular, X satisfies M-approximation if $m_i < \infty$ for all $i = 1, \dots, n$.

The special case of Corollary 1.1.8 where (K, C) is the function field of a curve over an algebraically closed field of characteristic 0 has been proven independently in a recent work by Chen, Lehmann and Tanimoto [CLT24], under the additional assumption that $m_i < \infty$ for all $i = 1, \dots, n$. They also obtain analogues of this result for other Campana pairs $(X, D_{\mathbf{m}})$ which are "Campana rationally connected". The method they use relies on logarithmic geometry, and differs from the approach taken in this thesis.

We also study failures of the $\mathcal{M}$-Hilbert property on split toric varieties. The main result in this direction is Theorem 3.3.5, which gives general sufficient conditions for the $\mathcal{M}$-Hilbert property to fail, and gives a measure of how badly it fails. It also gives a precise characterisation for Zariski density of the set of $\mathcal{M}$-points. As a consequence of the theorem it follows that over global fields, M-approximation is equivalent to the $\mathcal{M}$-Hilbert property.

Corollary 1.1.9. *Let (K, C) be a global field, let $B \subset C$ be a nonempty open set and set $T = \Omega_K \setminus B$. Let (X, M) be a toric pair where X is a normal complete split toric variety over K with toric integral model $(\mathcal{X}, \mathcal{M})$ over B. Then (X, M) satisfies M-approximation off T if and only if the $\mathcal{M}$-Hilbert property over B is satisfied, meaning that $(\mathcal{X}, \mathcal{M})(B)$ is not thin.*

If $T \neq \emptyset$, then the same holds for any integral model $(\mathcal{X}, \mathcal{M})$ over B such that $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$.

1.1.5 Darmon points and root stacks

In the final section on M-approximation, we elucidate the relation between Darmon points and root stacks, and we relate the conditions in Theorem 1.1.3 to the fundamental group of the associated root stack. Proposition 3.5.2 shows that outside of the boundary, integral points on the root stack $(\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}})$ are the same as Darmon points on $\mathcal{X}$. We also prove in Proposition 3.5.5 that the pair (X, M) corresponding to the Darmon points satisfies M-approximation if and only if the root stack $(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation, as studied in [Chr20; San23b].

In Lemma 3.5.8 we compute the étale fundamental group of a toric root stack, and show that it coincides with the profinite completion of the group N/N_M considered in Theorem 1.1.3. By combining the lemma with Theorem 1.1.3, we obtain the following characterisation of strong approximation for split toric root stacks, which generalizes Corollary 1.1.6 from toric varieties $V \subset X$ to toric root stacks $(X, \sqrt[\mathbf{m}]{D}) \to X$.

Corollary 1.1.10. *Let X be a smooth split toric variety over a PF field (K, C) of characteristic 0, let $D_1, \ldots, D_n$ be the the torus-invariant prime divisors on X, let $m_1, \ldots, m_n \in \mathbb{N}^* \cup \{\infty\}$ and let $D_{\mathbf{m}}$ be the corresponding Campana divisor as in Definition 2.1.19. Let $T \subset \Omega_K$ be a finite nonempty set of places and let $\overline{K}$ be an algebraic closure of K. Then*

1. *$(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation off T if $\pi_1(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}})$ is finite and $|\pi_1(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}})| \in \rho(K, C)$. The converse also holds if* $\mathrm{Pic}(C)$ *is finitely generated.*
2. *$(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation if and only if $(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}})$ is simply connected and $\mathcal{O}(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}}) = \overline{K}$.*

1.2 $\mathcal{M}$-points of bounded height

For the remainder of the thesis, we focus on counting $\mathcal{M}$-points. For this we restrict to rationally connected pairs, as we will define in Chapter 4. In Conjecture 1.2.2, we propose an asymptotic formula for the number of $\mathcal{M}$-points of bounded height, for any smooth, proper and rationally connected pair (X, M) over a number field. This conjecture generalizes both Manin's conjecture on rational points of bounded height [FMT89; Pey95; LST22], as well as its extension to Campana points [PSTVA21, Conjecture 1.1] as formulated by Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

Conjecture 1.2.2 is formulated using the Picard group of the pair, which we introduce and study in Chapter 4. In Chapter 5, we will then prove our conjecture on $\mathcal{M}$-points for toric pairs over $\mathbb{Q}$.

1.2.1 Heights and Manin's conjecture

A height on an algebraic variety X over a number field K is a function

$$H\colon X(K) \to \mathbb{R}_{>0},$$

which measures the "complexity" of a rational point. Heights are a very important tool in the study of rational points, used to prove many results in arithmetic geometry.

For instance, they are used in the proof of the aforementioned Mordell-Weil theorem [Wei28] and the Mordell Conjecture [Fal83]. The most studied height function is the Weil height on projective space. For a rational point $P = (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}(K)$, the Weil height is the product

$$H(P) = \prod_{v \in \Omega_K} \max(|x_1|_v, \ldots, |x_n|_v),$$

where Ω_K is the set of places of K and $|\cdot|_v$ is the v-adic norm on K. Thus for a given variety X with a given embedding $X \subset \mathbb{P}^{n-1}$, we get a height function on K by restricting the Weil height to X. More generally, given any line bundle L on X with an adelic metrization $\mathcal{L}$, we obtain a height

$$H_{\mathcal{L}} \colon X(K) \to \mathbb{R}_{>0}$$

on X as defined in [Pey95, §1.3]. Manin's conjecture gives a prediction for the number of rational points of bounded height on a rationally connected variety (such as a Fano variety). We recall the most recent version of the conjecture, which is given for example in [LST22, Conjecture 1.2].

Conjecture 1.2.1. *[Manin's conjecture] Let X be a proper smooth rationally connected variety over a number field K and assume that $X(K)$ is not a thin set. Then for every big and nef divisor class L with an adelic metrization $\mathcal{L}$, there exists a thin set $Z \subset X(K)$ such that*

$$\#\{P \in X(K) \setminus Z \mid H_{\mathcal{L}}(P) \leq B\} \sim cB^{a(X,L)}(\log B)^{b(K,X,L)-1} \quad \text{as } B \to \infty,$$

where $c > 0$ is a constant, $a(X, L)$ is the infimum of all rational numbers a such that $aL + K_X$ is an effective $\mathbb{Q}$-divisor class and $b(K, X, L)$ is the codimension of the minimal face of the pseudo-effective cone containing $a(X, L)L + K_X$.

As an important special case, the conjecture implies that on a Fano variety there exists a thin set Z such that the number of points $P \in X(K) \setminus Z$ with anti-canonical height $H_{-\mathcal{K}_X}(P) \leq B$ is asymptotic to

$$cB(\log B)^{\operatorname{rank}\operatorname{Pic}(X)-1}$$

as $cB \to \infty$, where $-\mathcal{K}_X$ is any metrization of the anti-canonical divisor class $-K_X$.

Manin's conjecture was first formulated and studied in 1989 and 1990 by Manin, Batyrev, Tschinkel and Franke [FMT89; BM90]. Peyre [Pey95] further contributed to the conjecture by giving a conjectural value for the constant c. More recently Lehmann, Sengupta and Tanimoto [LST22] have formulated a prediction for the thin set Z that has to be excluded.

1.2.2 An generalisation of Manin's conjecture for $\mathcal{M}$-points of bounded height

Now we will formulate a version of Manin's conjecture for $\mathcal{M}$-points. Let (X, M) be a smooth, proper and rationally connected pair over a number field K, as defined in Section 4.3. For such a pair, we introduce its Picard group $\operatorname{Pic}(X, M)$ along with a natural group homomorphism $\mathrm{pr}_M^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X,M)$, as well as its canonical divisor class $K_{(X,M)} \in \mathrm{Pic}(X,M)$. Using these notions, we define the Fujita invariant $a((X,M),L)$ and the b-invariant $b(K,(X,M),L)$ for pairs analogously to the respective invariants for varieties as in [LST22]. We use these geometric invariants to give an asymptotic formula for the number of $\mathcal{M}$-points of bounded height.

Let $\mathcal{L} = (L, \|\cdot\|)$ be an adelically metrized line bundle on X. For a subset $A \subset X(K)$ and an integer B, consider the counting function

$$N(A,\mathcal{L},B) = \#\{P \in A \mid H_{\mathcal{L}}(P) \leq B\}.$$

Fix a finite set S of places of K including all infinite places, and let $\mathcal{O}_S \subset K$ be the ring of S-integers.

Conjecture 1.2.2. *Let (X,M) be a smooth proper pair over a number field K such that (X,M) is rationally connected, and let $(\mathcal{X},\mathcal{M})$ be an integral model of (X,M) over $\mathcal{O}_S$. Assume furthermore that $(\mathcal{X},\mathcal{M})(\mathcal{O}_S) \subset X(K)$ is Zariski dense in X. Then for every big and nef divisor class L with an adelic metrization $\mathcal{L}$, there exists a thin set $Z \subset X(K)$ such that*

$$N((\mathcal{X},\mathcal{M})(\mathcal{O}_S) \setminus Z, \mathcal{L}, B) \sim cB^{a((X,M),L)}(\log B)^{b(K,(X,M),L)-1} \qquad \text{as } B \to \infty,$$

where $a((X,M),L)$ and $b(K,(X,M),L)$ are the Fujita invariant and the b-invariant as in Definition 4.2.14 and c is a constant.

Conjecture 1.2.2 can be directly seen to be a generalization of Manin's conjecture, formulated in Conjecture 1.2.1. It is not as straightforward to show that the conjecture generalizes its analogue for Campana points, formulated in [PSTVA21, Conjecture 1.1], but in Section 4.4 we show that the invariants in the two conjectures agree with each other. In particular, Conjecture 1.2.2 is also compatible with [CLT$^+$25, Conjecture 8.3], as that conjecture only differs from [PSTVA21, Conjecture 1.1] in the prediction of the leading constant c. Aside from implying the conjecture [PSTVA21, Conjecture 1.1] on Campana points, Conjecture 1.2.2 also predicts the asymptotic growth for the number of Darmon points and weak Campana points of bounded height, for which no such predictions exist in the literature. Weak Campana points of bounded height were discussed in [PSTVA21], but the authors of that article did not give any prediction for their asymptotic growth.

Remark 1.2.3. By Corollary 4.4.13, Conjecture 1.2.2 implies that for any Campana pair $(X, D_{\mathbf{m}})$, the number of Darmon points of bounded height has the same asymptotic growth as the number of Campana points of bounded height, up to possibly differing leading constants. For weak Campana points, the asymptotic growth is similar to Campana points as their Fujita invariants agree, but the exponent on the logarithm tends to be larger, as shown in Proposition 4.4.10.

Remark 1.2.4. As we will see in Section 3.5, Darmon points correspond to integral points on the corresponding root stack. Conjecture 1.2.2 therefore gives a prediction for the asymptotic number of integral points of bounded height on a root stack $(\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}})$, where the height is induced by any metrized big and nef line bundle on X. This is related to the conjecture by Ellenberg, Satriano and Zureick-Brown on the number of rational points of bounded height on stacks [ESZ23, Conjecture 4.14] and its generalization by Darda and Yasuda [DY24, Conjecture 9.16]. Their conjectures use different heights however: the heights we consider are what Darda and Yasuda call an unstable height, while their conjecture uses stable heights instead.

Remark 1.2.5. In contrast to Manin's conjecture, as we have formulated it in Conjecture 1.2.1, and its analogue for Campana points [PSTVA21, Conjecture 1.1], we do not assume that the set of $\mathcal{M}$-points in Conjecture 1.2.2 is not thin. Instead, we only require that it is Zariski dense. The reason for this is that Theorem 1.2.7, introduced in the next section, implies the conclusion of Conjecture 1.2.2 for every proper toric pair, even though there are many such pairs (X, M) for which the set of $\mathcal{M}$-points is thin by Theorem 3.3.5. Furthermore, there are no known examples of rationally connected varieties with a thin, but nonempty, set of rational points. Any such example would contradict an open conjecture by Colliot-Thélène [Ser08, Conjecture 3.5.8], [Col88].

Remark 1.2.6. In the analogue of Manin's conjecture for Campana points, formulated in [PSTVA21, Conjecture 1.2], the authors additionally assume that the integral model $\mathcal{X}$ is regular, which we do not assume here. The main reason for imposing such a restriction in their paper is to give a prediction for the leading constant c appearing in their conjecture. As we do not provide a prediction for this constant c appearing in Conjecture 1.2.2, we do not impose this condition.

1.2.3 $\mathcal{M}$-points of bounded height on toric varieties

In Chapter 5 we will show that Conjecture 1.2.2 is true for any smooth proper toric pair over $\mathbb{Q}$, i.e. for a smooth pair (X, M) where X is a smooth split toric variety over $\mathbb{Q}$ and the chosen divisors $D_1, \dots, D_n$ are torus-invariant. In this chapter, we will always take the integral model $(\mathcal{X}, \mathcal{M})$ to be the toric integral model as in Definition 3.1.1. Furthermore, for a divisor class $L \in \operatorname{Pic}(X)$, we let $\mathcal{L}$ be the metrized line bundle obtained by equipping L with the toric metric as in [BT96, Theorem 2.1.6] and we let $H_{\mathcal{L}}$ be the corresponding height on X, which we will recall in Section 5.2.1. For a positive integer S and any real number B, we consider the counting function

$$N_{(X,M),L,S}(B) := N((\mathcal{X}, \mathcal{M})(\mathbb{Z}[\tfrac{1}{S}]) \cap U(\mathbb{Q}), \mathcal{L}, B), \tag{1.2.1}$$

where U is the dense torus in X. The main result of this section is the following theorem, which implies Conjecture 1.2.2 for toric pairs over $\mathbb{Q}$.

Theorem 1.2.7. *Let (X, M) be a smooth proper toric pair over $\mathbb{Q}$ with toric integral model $(\mathcal{X}, \mathcal{M})$ over $\mathbb{Z}$ and let $L \in \operatorname{Pic}(X)$ be a big and nef divisor class. Then there exists $\theta > 0$ and a polynomial Q of degree $b(\mathbb{Q}, (X, M), L) - 1$ such that*

$$N_{(X,M),L,S}(B) = B^{a((X,M),L)} \left(Q(\log B) + O\left(B^{-\theta}\right)\right).$$

Furthermore, if L is adjoint rigid with respect to (X, M), then the leading coefficient of Q is explicitly given in Theorem 5.2.5 as well as in Theorem 5.2.10.

Here the condition that L is adjoint rigid means that the class $a((X, M), L) \operatorname{pr}_M^* L + K_{(X,M)}$ is represented by an unique $\mathbb{Q}$-divisor on (X, M), see Definition 4.2.20. This theorem is a special case of

Theorem 5.2.5, which also allows the pair to be quasi-proper as in Definition 4.2.21, rather than proper. To prove Theorem 1.2.7 for divisors which are not adjoint rigid with respect to (X, M), we need to consider quasi-proper pairs to show that Q has degree equal to, rather than at most, $b(\mathbb{Q}, (X, M), L) - 1$.

Theorem 1.2.7 implies [PSTVA21, Conjecture 1.1] for split toric varieties over $\mathbb{Q}$, including the conjecture for the leading constant if L is adjoint rigid. This generalizes the results by Pieropan and Schindler [PS24a, Theorem 1.2] to heights corresponding to divisors different from the log-anticanonical divisors, and improves on their error term. The theorem is proved using the universal torsor method, as developed by Salberger [Sal98]. The proof proceeds along the lines of de la Bretèche's proof [dlBre01a] of Manin's conjecture for split toric varieties with the anticanonical height, together with Salberger's computation of the leading constant [Sal98, Section 11]. Besides generalizing their results to $\mathcal{M}$-points, it generalizes their proofs to handle other heights than the anticanonical height.

Therefore, Theorem 1.2.7 is of interest even in the classical setting of rational points, as it improves on the original proof of Manin's conjecture for toric varieties by Batyrev and Tschinkel [BT96, Corollary 1.5] by providing a good control of the error term. In the classical setting of rational points, Theorem 1.2.7 and its proof are similar to [Ess07, Theorem 1], where heights coming from other metrizations are considered. We finish the section by giving a few examples to illustrate Theorem 1.2.7. In these examples, we will consider weak Campana points.

Example 1.2.8. Theorem 1.2.7 implies that

$$\#\left\{(x:y:z)\in\mathbb{P}^2(\mathbb{Q})\;\middle|\;\begin{array}{l}x,y,z\in\mathbb{Z}\setminus\{0\},\ \gcd(x,y,z)=1,\\ xyz \text{ is squareful, } \max(|x|,|y|,|z|)\le B\end{array}\right\} = B^{3/2}(Q(\log B)+O(B^{-\theta}))$$

as $B \to \infty$, where $\theta > 0$ is a constant and Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}} \left(1-p^{-1}\right)^6 \left(\frac{1-p^{-3/2}}{\left(1-p^{-1/2}\right)^3} - 3p^{-1/2}\right) \approx 0.862.$$

See Example 5.2.6 for the derivation of this result.

Example 1.2.9. More generally, Theorem 1.2.7 implies that for any positive integers m and n,

$$\#\left\{(x_1:\cdots:x_n)\in\mathbb{P}^{n-1}(\mathbb{Q})\;\middle|\;\begin{array}{l}x_1,\ldots,x_n\in\mathbb{Z}\setminus\{0\},\ \gcd(x_1,\ldots,x_n)=1,\\ \prod_{i=1}^n x_i \text{ is } m\text{-full, } \max(|x_1|,\ldots,|x_n|)\le B\end{array}\right\} = B^{n/m}(Q(\log B)+O(B^{-\theta}))$$

as $B \to \infty$, where $\theta > 0$ is a constant and Q is a polynomial of degree

$$\binom{m+n-1}{n-1} - \binom{m-1}{n-1} - n.$$

We will derive this result in Example 5.2.8.

Remark 1.2.10. In [Str22, Theorem 1.1], Streeter derived a very similar asymptotic formula for a related counting problem. Assume that $\gcd(n, m) = 1$ or n is prime. Under this assumption, he shows that for any given norm form N_ω for a Galois extension L/K of degree n,

$$\#\left\{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}(K) \;\middle|\; \begin{array}{l} x_1, \ldots, x_n \in \mathbb{Z},\ \gcd(x_1, \ldots, x_n) = 1, \\ N_\omega(x_1, \ldots, x_n) \text{ is } m\text{-full}, \\ \max(|x_1|, \ldots, |x_n|) \leq B \end{array}\right\} \sim cB^{m/n}(\log B)^{b(n,m)-1}$$

as $B \to \infty$, where $c > 0$ is a constant and

$$b(n, m) = \frac{1}{n}\left(\binom{m+n-1}{n-1} - \binom{m-1}{n-1}\right).$$

In particular, we find

$$\deg(Q) = n(b(n, m) - 1),$$

where Q is the polynomial in Example 1.2.9.

1.3 Notation and preliminaries

1.3.1 Natural numbers

We use the convention that the set of natural numbers $\mathbb{N}$ contains 0 and we write $\mathbb{N}^*$ for the set of nonzero natural numbers. We also define the set of extended natural numbers $\overline{\mathbb{N}} := \mathbb{N} \sqcup \{\infty\}$ as the one point compactification of the discrete space $\mathbb{N}$. We use the convention that $\frac{1}{\infty} = 0$. The topology on $\overline{\mathbb{N}}$ is the topology such that the map $\overline{\mathbb{N}} \to \mathbb{R}$ given by $n \mapsto \frac{1}{n+1}$ is a homeomorphism onto its image. We extend the greatest common divisor function to allow its arguments to lie in $\overline{\mathbb{N}}$ by setting $\gcd(\infty, a_1, \ldots, a_n) = \gcd(a_1, \ldots, a_n)$ and $\gcd(\infty) = 0$.

1.3.2 Algebra and analysis

We typically denote vectors using boldface and write their components using a normal face together with an index. For example, we may write $\mathbf{s} = (s_1, \ldots, s_n)$ for a vector in $\mathbb{R}^n$. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we write $\mathbf{a} > \mathbf{b}$ if $a_i > b_i$ for all $i = 1, \ldots, n$. We also denote the i-th basis vector of $\mathbb{R}^n$ by $\mathbf{e}_i$.

For an abelian group G, we write $G_\mathbb{Q} = G \otimes_\mathbb{Z} \mathbb{Q}$ and $G_\mathbb{R} = G \otimes_\mathbb{Z} \mathbb{R}$ for its base change to $\mathbb{Q}$ and $\mathbb{R}$, respectively. For a symbol D we write $\mathbb{Z}(D)$ for the group isomorphic to $\mathbb{Z}$ with generator D, and similarly we write $\mathbb{Q}(D) \cong \mathbb{Q}$ for the vector space with generator D.

The logarithm log refers to the natural logarithm. For two nonnegative functions f, g the notation $f(x) \ll g(x)$ means that there exists a constant c such that $f(x) \leq cg(x)$ for all x in the common domain of the functions f and g.

1.3.3 Geometry

For a field k we write $\overline{k}$ for a choice of an algebraic closure. All schemes are taken to be separated. For a scheme X over a base scheme S and a morphism $S' \to S$ of schemes, we denote the base change by S' as $X_{S'} := X \times_S S'$. If $S' = \operatorname{Spec} R$, we also write X_R in place of $X_{S'}$.

Given a $\mathbb{Q}$-Weil divisor $D = \sum_i a_i D_i$ on X we define its floor as $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$. If D is an effective Cartier divisor on X, then we will routinely identify it with the closed subscheme of X defined by the sheaf of ideals $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. If D_1 and D_2 are Cartier divisors, then the closed subscheme $D_1 + D_2$ is defined by the ideal sheaf $\mathcal{I} = \mathcal{O}_X(-(D_1 + D_2)) \subset \mathcal{O}_X$.

We define a variety over a field k to be a separated scheme of finite type over k, and a curve to be an integral variety of dimension 1. Note that curves are defined to be integral and separated of finite type, but not necessarily geometrically integral.

If k is a topological field and X is a variety over k, then $X(k)$ is equipped with the induced topology. This is the topology such that for any affine open subvariety $U \subset X$ with a closed embedding $U \to \mathbb{A}^n_k$ the map $U(k) \to k^n$ is a homeomorphism onto its image.

All fundamental groups considered in this thesis are étale fundamental groups.

1.3.4 PF fields

We now introduce PF fields, based on a course taught by Artin at Princeton in 1950/51, of which the lecture notes are found in [Art67, Chapter 12]. See Remark 1.3.4 for the etymology of the term.

Definition 1.3.1. A *PF field* is defined to be a pair (K, C) where either

- K is a number field, the function field of $C = \operatorname{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K, or
- K is the function field of a regular projective curve C over a field k.

We call K a *global field* if K is a number field or k is finite.

Remark 1.3.2. The scheme C is specified in order to give a good notion of a place of K if K is a function field. For example if $k = l(C')$ is a function field of some curve C' over a field l and $K = k(C)$, then the curve C cannot be recovered from the field K alone. This issue does not arise if k is finite or when the embedding $k \to K$ is specified.

Note that every finitely generated field extension K/k of transcendence degree 1 over a field k is naturally a PF field. Each such field is the function field of an affine curve over k, which we can compactify and normalize to obtain a regular projective curve C. This curve is the unique regular projective curve C with $K = k(C)$, since any birational map $C \to C'$ to a regular projective curve is a morphism and therefore an isomorphism.

Remark 1.3.3. Note that while the curve C is regular, it need not be geometrically connected nor geometrically reduced over k. For example, we can consider $C =$

$\mathbb{P}^1_k \times_k \operatorname{Spec} l$ for a finite separable extension l/k or a finite inseparable extension l/k, respectively. The former subtlety disappears if we replace the base field k with its algebraic closure k' in K, since C is a geometrically connected curve over k'. In particular, if k is perfect, then C will be a geometrically integral curve over k'. However, C need not be geometrically reduced over k' without this assumption, as shown by the curve

$$C = \{sx^p + ty^p + z^p = 0\} \subset \mathbb{P}^2_k$$

over the field $k = \mathbb{F}_p(s,t)$, where p is a prime number. Furthermore, even if C is geometrically integral over k, it need not be smooth as shown by the curve

$$C = \{tx^p + z^{p-1}y + y^p = 0\} \subset \mathbb{P}^2_k$$

over the field $k = \mathbb{F}_p(t)$, where $p > 2$ is a prime number.

Note that if $B \subset C$ is an open subscheme, then B is affine, unless K is a function field and $B = C$.

We use the convention that a discrete valuation on K contains 1 in its image. If K is a number field, then a *finite place* of K is a discrete valuation on K. If K is a function field, then a *finite place* of K is a discrete valuation on K which is trivial on k. We denote the set of finite places of K by $\Omega_K^{<\infty}$. There is a natural bijection between the closed points on C and $\Omega_K^{<\infty}$, and we will thus routinely identify finite places and closed points. Given a finite place v of K, its degree is the degree of the associated closed point, and define the absolute value on K induced by v as

$$|a|_v = p^{-\deg(v)v(a)}.$$

Here, p is the characteristic of the residue field k_v if $\operatorname{char}(k_v) > 0$, and $p = 2$ if $\operatorname{char}(k_v) = 0$.

For a number field, an *infinite place* is an embedding $v\colon K \to \mathbb{C}$, where conjugate embeddings into $\mathbb{C}$ are identified. We denote the set of infinite places of K by Ω_K^∞. An infinite place v is *real* if it factors through an embedding $K \to \mathbb{R}$ and *complex* otherwise. An infinite place v induces an absolute value on K by

$$|a|_v = |v(a)|^e,$$

where $|\cdot|$ is the standard absolute value on $\mathbb{C}$ and $e = 1$ if v is real and $e = 2$ if v is complex. If K is a function field, we set $\Omega_K^\infty = \emptyset$. For any PF field K we define the set of *places* on K to be

$$\Omega_K = \Omega_K^{<\infty} \sqcup \Omega_K^\infty.$$

Remark 1.3.4. The term PF field stands for Product Formula field, named after the formula

$$\prod_{v \in \Omega_K} |x|_v = 1$$

for all $x \in K^\times$.

For a place $v \in \Omega_K$, we denote by K_v the completion of K with respect to the absolute value $|\cdot|_v$. This field is locally compact if and only if K is a global field. If v is a finite place, we set

$$\mathcal{O}_v = \{x \in K_v \mid |x|_v \leq 1\}.$$

For an infinite place v, we simply set $\mathcal{O}_v = K_v$. If $\mathcal{X}$ is a scheme over an open subscheme $B \subset C$, then for a place $v \in B$, we write $\mathcal{X}_v = \mathcal{X} \times_B \operatorname{Spec} \mathcal{O}_v$.

1.3.5 Restricted products and adeles

Definition 1.3.5. Let I be an index set, and for each $i \in I$, let X_i be a topological space with a subspace $U_i \subset X_i$, which is not necessarily open. Then the underlying set of the *restricted product* of these spaces is

$$\prod_{i\in I}(X_i, U_i) := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; x_i \in U_i \text{ for all but finitely many } i \in I \right\}.$$

This set is given the finest topology such that for all finite sets $J \subset I$, the inclusion map

$$\pi_J \colon \prod_{i\in J} X_i \times \prod_{i\in I\setminus J} U_i \hookrightarrow \prod_{i\in I}(X_i, U_i)$$

is continuous.

If $J' \subset J$ are finite subsets of I, then the map $\prod_{i\in J'} X_i \times \prod_{i\in I\setminus J'} U_i \to \prod_{i\in J} X_i \times \prod_{i\in I\setminus J} U_i$ is a continuous map. Thus for any finite subset $J \subset I$ there is a natural homeomorphism

$$\prod_{i\in I}(X_i, U_i) \cong \prod_{i\in J} X_i \times \prod_{i\in I\setminus J}(X_i, U_i).$$

We will now consider the topological properties of two types of inclusions between restricted products.

Proposition 1.3.6. *For $i \in I$, let X_i be a topological space with subspaces $Z_i \subset Y_i \subset X_i$. Then*

1. *if $\prod_{i\in I}(X_i, Z_i) \neq \emptyset$, the natural inclusion*

$$\prod_{i\in I}(X_i, Z_i) \hookrightarrow \prod_{i\in I}(X_i, Y_i)$$

is continuous and has dense image.

2. *The natural inclusion $\prod_{i\in I}(Y_i, Z_i) \hookrightarrow \prod_{i\in I}(X_i, Z_i)$ is a topological embedding and it is open if $Y_i \subset X_i$ is open for all $i \in I$.*

Proof. For a finite set $J \subset I$, we define $(X, Y)_J := \prod_{i\in J} X_i \times \prod_{i\in I\setminus J} Y_i$ and we define $(X, Z)_J$ and $(Y, Z)_J$ similarly. We will first prove the first statement. By the definition of the restricted product topology, a subset $V \subset \prod_{i\in I}(X_i, Y_i)$ is open if and only if for every finite subset $J \subset I$, $V \cap (X, Y)_J$ is open in $(X, Y)_J$. Thus if V is an open subset of $\prod_{i\in I}(X_i, Y_i)$, then $V \cap (X, Z)_J$ is open in $(X, Z)_J$ so the inclusion is continuous.

To show that the map has dense image, we show that $V \cap \prod_{i\in I}(X_i, Z_i) \neq \emptyset$ for every nonempty subset $V \subset \prod_{i\in I}(X_i, Y_i)$. Let $J \subset I$ be such that $(X, Z)_J \neq \emptyset$. For any finite subset $J \subset I$ such that $V \cap (X, Y)_J \neq \emptyset$, the set $V \cap (X, Y)_J$ contains an nonempty open subset $\prod_{i\in I} V_i$, with V_i open in X_i if $i \in J$, V_i open in Y_i if $i \in I \setminus J$ and $V_i = Y_i$ for all but finitely many $i \in I$, by basic properties of the product topology. Let $J' \subset I$ be the finite subset of $i \in I$ for which $V_i \neq Y_i$ or $Y_i \neq \emptyset$, then $V \cap (X, Z)_{J'} \neq \emptyset$ and the map therefore has dense image.

Now we will prove the second statement. The image $\mathrm{im}(\iota)$ of $\iota\colon \prod_{i\in I}(Y_i, Z_i) \hookrightarrow \prod_{i\in I}(X_i, Z_i)$ with the subspace topology has the finest topology such that every continuous map $A \to \prod_{i\in I}(X_i, Z_i)$ with set-theoretic image in $\mathrm{im}(\iota)$ factors continuously through $\mathrm{im}(\iota)$. Therefore if ι is continuous, then it is an embedding. If $V \subset \prod_{i\in I}(X_i, Z_i)$ is open, then $V \cap (X, Z)_J$ is open in $(X, Z)_J$ for every finite subset $J \subset I$, and since $(Y, Z)_J$ is a subspace of $(X, Z)_J$, $V \cap (Y, Z)_J$ is open in $(Y, Z)_J$. Thus the inclusion map is continuous. If furthermore $Y_i \subset X_i$ is open for all $i \in I$, then $V \cap \prod_{i\in I}(X_i, Z_i)$ is open, so the inclusion map is an open map. □

Using this construction, we define the ring of adeles.

Definition 1.3.7. Let (K, C) be a PF field and let T be a finite set of places. The *ring of adeles of K prime to T* is the topological K-algebra

$$\mathbf{A}_K^T = \prod_{v\in\Omega_K\setminus T} (K_v, \mathcal{O}_v).$$

For a nonempty open subset $B \subset C$ the *ring of B-integral adeles prime to T* is the topological K-algebra

$$\mathbf{A}_B^T = \prod_{v\in B} \mathcal{O}_v \times \prod_{v\in\Omega_K\setminus B} K_v.$$

The following proposition will be used in Proposition 3.5.2 to relate adelic Darmon points to adelic points on the associated root stack.

Proposition 1.3.8. *Let (K, C) be a PF field, let T be a finite set of places and let $B \subset C$ be an open subset. Then every finitely generated ideal in $\mathbf{A}_K^T$ or $\mathbf{A}_B^T$ is principal. In particular,* $\mathrm{Pic}(\mathbf{A}_K^T) = \mathrm{Pic}(\mathbf{A}_B^T) = 0$.

Proof. We will prove the statement for $\mathbf{A}_K^T$ and note that the statement for $\mathbf{A}_B^T$ follows analogously. If we have an ideal $I = ((a_v)_{v\in\Omega_K\setminus T}, (b_v)_{v\in\Omega_K\setminus T})$, then we can for every finite place $v \in \Omega_K^{<\infty} \setminus T$ consider $t_v, s_v \in \mathcal{O}_v^\times$ such that $v(c_v) = \min(v(a_v), v(b_v))$, where $c_v = t_v a_v + s_v b_v$. Then $I = ((c_v)_{v\in\Omega_K\setminus T})$ and therefore I is principal. Thus induction on the number of generators shows that every finitely generated ideal is principal. By [Stacks, Tag 0B8N], invertible ideals are finitely generated, hence $\mathrm{Pic}(\mathbf{A}_K^T) = 0$. □

2. Pairs and $\mathcal{M}$-points

In this chapter we introduce pairs and $\mathcal{M}$-points and we will illustrate these concepts with many examples. In Chapter 2, we will introduce the M-approximation and we prove general results about pairs satisfying this property. In Chapter 3, we continue the study of M-approximation in the special setting of toric pairs.

2.1 $\mathcal{M}$-points

In this section we introduce $\mathcal{M}$-points, generalizing the notions of integral points and Campana points. We fix a PF field (K, C) and an open subscheme $B \subset C$.

2.1.1 Integral models of pairs

First we define pairs and their integral models.

Definition 2.1.1. Let X be a scheme over a scheme B and let $(D_\alpha)_{\alpha\in\mathcal{A}}$ be a finite tuple of closed subschemes on X. Let $\mathfrak{M} \subset \overline{\mathbb{N}}^{\mathcal{A}}$ be a subset satisfying $(0, \dots, 0) \in \mathfrak{M}$ and such that for all $\mathbf{m} \in \mathfrak{M}$ the element $\mathbf{m}'$ defined by

$$m'_\alpha = \begin{cases} 0 & \text{if } m_\alpha \neq \infty, \\ \infty & \text{if } m_\alpha = \infty, \end{cases}$$

lies in $\mathfrak{M}$.
For such a set we let

$$M := ((D_\alpha)_{\alpha\in\mathcal{A}}, \mathfrak{M})$$

and we call (X, M) a *pair* over B , $\mathfrak{M}$ the *set of multiplicities*, and M the *parameter set*. We will call $\bigcup_{\alpha\in\mathcal{A}} D_\alpha$ the *boundary* of (X, M) and denote its complement in X by $U = X \setminus \bigcup_{\alpha\in\mathcal{A}} D_\alpha$. If $\mathcal{A} = \emptyset$, then we write $M = 0$ and we say that M is trivial.

The technical condition on $\mathfrak{M}$ is very mild and it will ensure that for any place $v \in \Omega_K$ the $\mathcal{M}$-points over $\mathcal{O}_v$ lie in the $\mathcal{M}$-points over K_v, as we will define in Definition 2.1.12. This definition generalizes the notion of Campana pairs given in [Cam11a, Définition 2.1], which we recover if $B = \operatorname{Spec}\mathbb{C}$, X is a normal variety, the D_α are Weil divisors and $\mathfrak{M}$ is chosen to encode the Campana condition found in Definition 2.1.19.

Remark 2.1.2. Note that the notion of a pair (X, M) is more general than the notion of an M-curve as studied by Darmon in [Dar97], even when we restrict X to be a curve. However, an M-curve can be naturally viewed as a pair, as we will see in Definition 2.1.19.

In the later chapters on toric varieties, the subschemes D_α will be divisors, but there are advantages to allowing them to be arbitrary closed subschemes, as we will see later in this chapter. As in the case of Campana orbifolds [Cam05; Abr09; PSTVA21], the points on the pair are only defined after a choice of an integral model, which we define as follows.

Definition 2.1.3. Let X be a proper variety over a PF field (K, C). A scheme $\mathcal{X}$ over B is an *integral model of* X *over* B if it is proper over B and its generic fiber is isomorphic to X over K.

Note that we do not require the integral models to be flat over B.

Definition 2.1.4. Given a pair (X, M) with X a proper variety over a PF field (K, C), an *integral model of* (X, M) *over* B is a pair $(\mathcal{X}, \mathcal{M})$, where $\mathcal{X}$ is an integral model of X over B and $\mathcal{M} = ((\mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \mathfrak{M})$, where for all $\alpha \in \mathcal{A}$, $\mathcal{D}_\alpha \subset \mathcal{X}$ is an integral model of D_α over B. We also say that $(\mathcal{X}, \mathcal{M})$ is a pair over B.

Note that we do not require the $\mathcal{D}_\alpha$ to be flat over B. Given an integral model over B, we can restrict it to open sets in B as follows.

Definition 2.1.5. Let (X, M) be a pair with integral model $(\mathcal{X}, \mathcal{M})$ over B. If $B' \subset B$ is a nonempty open subset, then we define the integral model $(\mathcal{X}, \mathcal{M})_{B'}$ over B' as

$$(\mathcal{X}, \mathcal{M})_{B'} = (\mathcal{X} \times_B B', \mathcal{M}_{B'}),$$

where $\mathcal{M}_{B'} = ((\mathcal{D}_\alpha \times_B B')_{\alpha\in\mathcal{A}}, \mathfrak{M})$.

In the literature on Campana points, such as [PSTVA21; NS24; MNS24], there is a canonical choice of an integral model $\mathcal{D}_\alpha$ of D_α by taking the Zariski closure and endowing it with the reduced scheme structure. We generalize this construction to allow D to be nonreduced.

Proposition 2.1.6. *Let X be a proper variety over K with integral model $\mathcal{X}$. If $D \subset X$ is a closed subscheme over K, then there exists a unique closed subscheme $\mathcal{D}^c \subset \mathcal{X}$ with generic fiber $\mathcal{D}^c_K = D$ such that the inclusion $\mathcal{D}^c \subset \mathcal{X}$ factors through every closed subscheme $\widetilde{\mathcal{D}} \subset \mathcal{X}$ with $\widetilde{\mathcal{D}}_K = D$.*

Proof. Consider the sheaf of ideals $\mathcal{I}$ on $\mathcal{X}$ defined as the kernel of the composition $\mathcal{O}_\mathcal{X} \to \mathcal{O}_X \to \mathcal{O}_D$ of $\mathcal{O}_\mathcal{X}$-algebras. This sheaf defines a closed subscheme $\mathcal{D}^c$ with $\mathcal{D}^c_K = D$. By construction D^c has the desired universal property. □

We will call $\mathcal{D}^c$ the *closure* of D in $\mathcal{X}$. The next proposition shows that the closure interacts very well with the structure of Cartier divisors.

Proposition 2.1.7. *Let X be a proper variety over a PF field (K, C) with integral model $\mathcal{X}$ over $B \subset C$. If $\mathcal{D}$ is an irreducible effective Cartier divisor on $\mathcal{X}$ with $\mathcal{D}_K = D \neq \emptyset$, then $\mathcal{D} = \mathcal{D}^c$. In particular, if $D_1, D_2 \subset X$ are subschemes such that their closures $\mathcal{D}^c_1, \mathcal{D}^c_2$ in $\mathcal{X}$ are effective Cartier divisors, then $(\mathcal{D}_1 + \mathcal{D}_2)^c$ is an effective Cartier divisor and*

$$\mathcal{D}^c_1 + \mathcal{D}^c_2 = (\mathcal{D}_1 + \mathcal{D}_2)^c$$

as subschemes of $\mathcal{X}$.

Proof. By construction, $\mathcal{D}^c$ is a closed subscheme of $\mathcal{D}$. Therefore, by [Stacks, Tag 0AGB], there exists a Cartier divisor $\mathcal{D}'$ on $\mathcal{X}$ such that $\mathcal{D}' \subset \mathcal{D}^c$ is an isomorphism outside codimension 2. Now [Stacks, Tag 02ON] implies that there exists a Cartier divisor $\mathcal{D}''$ such that $\mathcal{D} = \mathcal{D}' + \mathcal{D}''$. As $\mathcal{D}$ is irreducible, we find $\mathcal{D}_{\mathrm{red}} = \mathcal{D}''_{\mathrm{red}}$ or $\mathcal{D}'' = \emptyset$. If we write $D' = \mathcal{D}'_K$ and $D'' = \mathcal{D}''_K$, then we see $D = D' + D''$. However, since D and D' are isomorphic outside a codimension 2 subset, the codimension of D'' is at least 2 so $D'' = \emptyset$, and thus $\mathcal{D}'' = \emptyset$. This implies $\mathcal{D} = \mathcal{D}'$ and therefore $\mathcal{D} = \mathcal{D}^c$. The second part of the proposition now follows from the fact that $D_1^c + D_2^c$ is a Cartier divisor with generic fiber $D_1 + D_2$, so $(\mathcal{D}_1 + \mathcal{D}_2)^c = \mathcal{D}_1^c + \mathcal{D}_2^c$. $\square$

Using the above construction, an integral model of the variety X induces an integral model of the pair (X, M):

Definition 2.1.8. Given a pair (X, M) and an integral model $\mathcal{X}$ of X over B, the *integral model of* (X, M) *induced by* $\mathcal{X}$ is the pair $(\mathcal{X}, \mathcal{M}^c)$ over B, where $\mathcal{M}^c = ((\mathcal{D}^c_\alpha)_{\alpha \in \mathcal{A}}, \mathfrak{M})$.

Note that by spreading out (cf. [Poo17, §3.2]) any proper variety X over K has an integral model over some nonempty open subscheme $B \subset C$. Hence, any pair (X, M) over K has an integral model over such an open subscheme $B \subset C$.

2.1.2 Multiplicities and $\mathcal{M}$-points

Now we will define intersection multiplicities and $\mathcal{M}$-points. As before, we let X be a proper variety over a PF field (K, C) with integral model $\mathcal{X}$ over an open subscheme $B \subset C$. Let $v \in B$ be a closed point and let $P \in X(K_v)$. By the valuative criterion of properness, P lifts to an unique point $\mathcal{P} \in \mathcal{X}(\mathcal{O}_v)$. For a closed subscheme $\mathcal{D} \subset \mathcal{X}$ we consider the scheme theoretic intersection $\mathcal{P} \cap \mathcal{D}$, which is defined as the fiber product of $\mathcal{P}\colon \operatorname{Spec} \mathcal{O}_v \to \mathcal{X}$ and the closed immersion $i_{\mathcal{D}}\colon \mathcal{D} \hookrightarrow \mathcal{X}$:

$$\begin{tikzcd} \mathcal{P} \cap \mathcal{D} \arrow[r] \arrow[d, hook] & \mathcal{D} \arrow[d, hook, "i_{\mathcal{D}}"] \\ \operatorname{Spec} \mathcal{O}_v \arrow[r, "\mathcal{P}"] & \mathcal{X}. \end{tikzcd}$$

As base change preserves closed immersions it follows that $\mathcal{P} \cap \mathcal{D} = \operatorname{Spec}(\mathcal{O}_v / I)$ for an ideal $I \subset \mathcal{O}_v$. As $\mathcal{O}_v$ is a discrete valuation ring, $I = (0)$ or $I = (\pi^n)$ for some $n \in \mathbb{N}$, where π is a uniformizer of $\mathcal{O}_v$. As in [MNS24, Definition 2.4] we make the following definition.

Definition 2.1.9. The *(local) intersection multiplicity* $n_v(\mathcal{D}, \mathcal{P})$ is defined to be

$$n_v(\mathcal{D}, \mathcal{P}) = \begin{cases} n & \text{if } I = (\pi^n), \\ \infty & \text{if } I = (0). \end{cases}$$

Note in particular that $n_v(\mathcal{D}, \mathcal{P}) = \infty$ exactly if $\mathcal{P} \subset \mathcal{D}(\mathcal{O}_v)$.

This definition agrees with the classical notion of local intersection multiplicity: if $\mathcal{X}$ is a smooth surface over an algebraically closed field, $\mathcal{O}_v = \mathcal{O}_{\mathcal{C},p}$ is the local ring of a point p on a smooth curve $\mathcal{C} \subset \mathcal{X}$ and $\mathcal{D} \subset \mathcal{X}$ is a Cartier divisor, then $n_v(\mathcal{P}, \mathcal{D}) = (\mathcal{C} \cap \mathcal{D})_p$ is the local intersection multiplicity of $\mathcal{C}$ and $\mathcal{D}$ in p as defined in [Har77, Chapter V] unless $\mathcal{C} \subset \mathcal{D}$ in which case $n_v(\mathcal{D}, \mathcal{P}) = \infty$.

Example 2.1.10. If $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_v}$ and $\mathcal{D}_i$ is the i-th coordinate hyperplane, then given an integral point $\mathcal{P} = (a_0 : \cdots : a_n)$, with $a_i \in \mathcal{O}_v$ for all $i \in \{0, \dots, n\}$ and $v(a_i) = 0$ for some $i \in \{0, \dots, n\}$, the intersection multiplicity is just the valuation $n_v(\mathcal{D}_i, \mathcal{P}) = v(a_i)$.

The next proposition shows that the intersection multiplicity respects addition of Cartier divisors.

Proposition 2.1.11. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be Cartier divisors on $\mathcal{X}$ and let $\mathcal{P} \in \mathcal{X}(\mathcal{O}_v)$. Then*

$$n_v(\mathcal{D}_1 + \mathcal{D}_2, \mathcal{P}) = n_v(\mathcal{D}_1, \mathcal{P}) + n_v(\mathcal{D}_2, \mathcal{P}).$$

Proof. This follows from the equality

$$\begin{aligned}\mathcal{P}^*\mathcal{O}_X(-(\mathcal{D}_1 + \mathcal{D}_2))\mathcal{O}_Y &= \mathcal{P}^*(\mathcal{O}_X(-\mathcal{D}_1)\mathcal{O}_X(-\mathcal{D}_2))\mathcal{O}_Y \\ &= \mathcal{P}^*\mathcal{O}(-\mathcal{D}_1)\mathcal{O}_Y \cdot \mathcal{P}^*\mathcal{O}(-\mathcal{D}_2)\mathcal{O}_Y,\end{aligned}$$

of ideal sheaves on $Y = \operatorname{Spec} \mathcal{O}_v$, which implies the identity. $\square$

Given a PF field (K, C) and a pair $(\mathcal{X}, \mathcal{M})$ over $B \subset C$ and a finite place $v \in B$, we define the map

$$\operatorname{mult}_v \colon \mathcal{X}(\mathcal{O}_v) \to \overline{\mathbb{N}}^{\mathcal{A}}, \quad \mathcal{P} \mapsto (n_v(\mathcal{D}_\alpha, \mathcal{P}))_{\alpha \in \mathcal{A}}.$$

We also define, for a field extension L/K, the map

$$\operatorname{mult}_L \colon X(L) \to \{0, \infty\}^{\mathcal{A}}, \quad \mathcal{P} \mapsto (n_L(D_\alpha, P))_{\alpha \in \mathcal{A}},$$

where

$$n_L(D_\alpha, P) := \begin{cases} 0 & \text{if } P \notin D_\alpha(L), \\ \infty & \text{if } P \in D_\alpha(L) \end{cases}$$

indicates whether the point P lies in D_α. Using these notions we are finally ready to define $\mathcal{M}$-points.

Definition 2.1.12. Let (K, C) be a PF field, and let (X, M) be a pair over K with integral model $(\mathcal{X}, \mathcal{M})$ over an open subscheme $B \subset C$. For a field extension L/K, we set

$$(X, M)(L) = (\mathcal{X}, \mathcal{M})(L) = \{P \in X(L) \mid \operatorname{mult}_L(P) \in \mathfrak{M}\}.$$

For a finite place $v \in B$, the set of *v-adic $\mathcal{M}$-points on* $(\mathcal{X}, \mathcal{M})$ is defined as

$$(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) = \{\mathcal{P} \in \mathcal{X}(\mathcal{O}_v) \mid \operatorname{mult}_v(\mathcal{P}) \in \mathfrak{M}\}. \tag{2.1.1}$$

If $v \in \Omega_K \setminus B$, we set

$$(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) = (X, M)(K_v).$$

The set of *$\mathcal{M}$-points on* $(\mathcal{X}, \mathcal{M})$ *over* B is defined as the subset of $\mathcal{X}(B)$ satisfying Condition (2.1.1) at every place $v \in B$:

$$(\mathcal{X}, \mathcal{M})(B) = \{\mathcal{P} \in \mathcal{X}(B) \mid \operatorname{mult}_v(\mathcal{P}_v) \in \mathfrak{M} \text{ for all } v \in B\}.$$

Note that $\bigcup_{B\subset C}(\mathcal{X},\mathcal{M})(B) = (X,M)(K)$, where the union runs over all nonempty open subschemes B of C.

Definition 2.1.13. Let (X,M) be a pair over K with integral model $(\mathcal{X},\mathcal{M})$ be a pair over a scheme B. We define $\mathfrak{M}_{\mathrm{fin}} = \mathfrak{M}\cap\mathbb{N}^{\mathcal{A}}$ and we define M_{fin} and $\mathcal{M}_{\mathrm{fin}}$ by replacing $\mathfrak{M}$ by $\mathfrak{M}_{\mathrm{fin}}$.

The points on $(\mathcal{X},\mathcal{M}_{\mathrm{fin}})$ are those in $(\mathcal{X},\mathcal{M})$ whose image does not lie in the boundary $\bigcup_{\alpha\in\mathcal{A}}\mathcal{D}_\alpha$.

Remark 2.1.14. If we assume that the subschemes $\mathcal{D}_\alpha$ are all Cartier divisors on $\mathcal{X}$, then we can give a different description of $(\mathcal{X},\mathcal{M})(B)$. Namely, if the image of a morphism $\mathcal{P}\colon B\to\mathcal{X}$ does not lie in the boundary $\bigcup_{\alpha\in\mathcal{A}}\mathcal{D}_\alpha$, then the intersection with $\mathcal{D}_\alpha$ is simply the pullback $\mathcal{P}\cap\mathcal{D}_\alpha = \mathcal{P}^*\mathcal{D}_\alpha$. So it follows that $(\mathcal{X},\mathcal{M}_{\mathrm{fin}})(B)$ is the set of points $\mathcal{P}\in\mathcal{X}(B)$ not contained in the boundary, such that there exist $\mathbf{m}_1,\dots,\mathbf{m}_k\in\mathfrak{M}_{\mathrm{fin}}$ and distinct prime divisors $\widetilde{\mathcal{D}}_1,\dots,\widetilde{\mathcal{D}}_k$ in B such that

$$\mathcal{P}^*\mathcal{D}_\alpha = \sum_{i=1}^{k} m_{i,\alpha}\widetilde{\mathcal{D}}_i$$

for all $\alpha\in\mathcal{A}$.

For many interesting choices of $\mathcal{M}$, the set of multiplicities $\mathfrak{M}$ is an open subset of $\overline{\mathbb{N}}^{\mathcal{A}}$. For example, this is the case for integral points, Campana points and strict Darmon points. In this case, the next proposition shows that the property of being a $\mathcal{M}$-point is open in $X(K_v)$.

Proposition 2.1.15. *Let X be a proper variety over a PF field (K,C) and let (X,M) be a pair with integral model $(\mathcal{X},\mathcal{M})$ over $B\subset C$. Then the map $\mathrm{mult}_v\colon X(K_v)\to\overline{\mathbb{N}}^{\mathcal{A}}$ is continuous for every finite place $v\in B$.*

Therefore, if $\mathfrak{M}\subset\overline{\mathbb{N}}^{\mathcal{A}}$ is an open (or closed) subset, then $(\mathcal{X},\mathcal{M})(\mathcal{O}_v)$ is an open (or closed) subset of $X(K_v)$.

Proof. It suffices to prove $n_v(\mathcal{D},-)\colon X(K_v)\to\overline{\mathbb{N}}$ is continuous for a single subscheme $\mathcal{D}$, as continuity is equivalent to continuity in all coordinates. Note that for $\mathcal{P}\in\mathcal{X}(\mathcal{O}_v) = X(K_v)$, the multiplicity $n_v(\mathcal{D},\mathcal{P})$ is the largest integer n_0 such that there exists a factorisation

$$\begin{tikzcd}
 & \mathcal{D}\times_{\mathcal{O}_v}\operatorname{Spec}\mathcal{O}_v/(\pi_v^{n_0}) \arrow[d] \\
\operatorname{Spec}\mathcal{O}_v/(\pi_v^{n_0}) \arrow[r] \arrow[ur, dotted] & \mathcal{X}\times_{\mathcal{O}_v}\operatorname{Spec}\mathcal{O}_v/(\pi_v^{n_0}),
\end{tikzcd}$$

where the horizontal and vertical homomorphisms are induced by $\mathcal{P}$ and by the inclusion morphism $i_{\mathcal{D}}\colon\mathcal{D}\to\mathcal{X}$, respectively. In particular, if two points $\mathcal{P},\mathcal{P}'\in\mathcal{X}(\mathcal{O}_v)$ have the same reduction modulo (π_v^n) for some integer n, either

$$n_v(\mathcal{D},\mathcal{P}) = n_v(\mathcal{D},\mathcal{P}') < n$$

or

$$\min\{n_v(\mathcal{D},\mathcal{P}), n_v(\mathcal{D},\mathcal{P}')\} \geq n.$$

Since the collection of open sets of the form

$$U(\mathcal{P},n) = \{\mathcal{P}' \in \mathcal{X}(\mathcal{O}_v) \mid \mathcal{P} \bmod \pi_v^n = \mathcal{P}' \bmod \pi_v^n \in \mathcal{X}(\mathcal{O}_v/\pi_v^n)\},$$

$\mathcal{P} \in X(K_v)$ and $n \in \mathbb{N}$, forms a basis for the topology on $\mathcal{X}(\mathcal{O}_v)$, it follows that $n_v(\mathcal{D}, -)$ is indeed continuous. Thus mult_v is a continuous map. □

2.1.3 $\mathcal{M}$-points over other schemes

The definition of $\mathcal{M}$-points over PF fields has a natural generalization to arbitrary schemes, which we will give in this section. We will use this generalization in Chapter 4 to define rational connectedness of a pair (X, M) over a field, for which we consider M-points on X over the projective line $\mathbb{P}^1$. Let X and Y be schemes over a base scheme S and let $P\colon Y \to X$ be a morphism over S. For every prime Cartier divisor v on Y and any closed subscheme $D \subset X$ we can define an analogue $n_v(D, P)$ of the local intersection multiplicity as given in Definition 2.1.9 by considering the fiber product $P \cap D := \mathrm{Spec}(\mathcal{O}_v) \times_X D$, where $\mathcal{O}_v$ is the local ring of Y along v, and the morphism $\mathrm{Spec}(\mathcal{O}_v) \to X$ is induced by the natural morphism $\mathrm{Spec}(\mathcal{O}_v) \to Y$. Since $\mathcal{O}_v$ is a discrete valuation ring, $P \cap D = \mathrm{Spec}\,\mathcal{O}_v$ or $P \cap D = \mathrm{Spec}(\mathcal{O}_v/I^n)$, where I is the maximal ideal of $\mathcal{O}_v$ and $n \in \mathbb{N}$.

Definition 2.1.16. The local intersection multiplicity $n_v(D, P)$ is defined to be

$$n_v(D,P) = \begin{cases} n & \text{if } I = (\pi^n), \\ \infty & \text{if } I = (0). \end{cases}$$

For any pair (X, M), we use this to define a multiplicity map

$$\mathrm{mult}_v\colon \mathrm{Hom}(Y, X) \to \overline{\mathbb{N}}^{\mathcal{A}},$$

given by $P \mapsto (n_v(D_\alpha, P))_{\alpha\in\mathcal{A}}$, extending the definition of the multiplicity map given in Section 2.1.2.

Definition 2.1.17. Let (X, M) be a pair over a scheme S and let Y be a scheme over S which is Noetherian, connected and regular. Assume furthermore that Y is not the spectrum of a field. Then a S-morphism $P\colon Y \to X$ is an *M-point over Y* if $\mathrm{mult}_v(P) \in \mathfrak{M}$ for all prime divisors v on Y. We denote the set of M-points over Y by $(X, M)(Y)$. If $Y = \mathrm{Spec}\,R$ for some ring R, then we also write $(X, M)(R) := (X, M)(Y)$.

Note that when (K, C) is a PF field, $S = B$ for some open subscheme $B \subset C$ and $Y = B$ or $Y = \mathrm{Spec}\,\mathcal{O}_v$ for a finite place $v \in B$, then this definition agrees with the original description given in Definition 2.1.12.

The assumption that Y is Noetherian and regular ensures that every divisor on Y is a sum of prime Cartier divisors by [Stacks, Tag 0BCP]. The reason why we do not allow Y to be the spectrum of a field in this definition is that such schemes do not have any prime divisors. For fields, we define M-points in the same way we did in Definition 2.1.12.

Remark 2.1.18. If we additionally assume that the set $\mathfrak{M}_{\text{fin}}$ is a monoid with topological closure $\mathfrak{M}$ and that the subschemes D_α are Cartier divisors, then the assignment

$$Y \mapsto (X, M)(Y)$$

is a functor (X, M) from the category of connected and regular Noetherian schemes over S to the category of sets. This follows from the following observation: if $f\colon Y' \to Y$ is a morphism of such schemes over S and $P\colon Y \to X$ is a morphism over S, then for every prime divisor v' on Y', we have

$$\operatorname{mult}_{v'}(P \circ f) = \sum_{v \text{ prime divisor on } Y} c_v \operatorname{mult}_v(P),$$

for $c_v \in \overline{\mathbb{N}}$. Here c_v is the multiplicity of v' in the pullback of v to Y' if $f(Y') \not\subset v$ and $c_v = \infty$ otherwise.

2.1.4 Examples of $\mathcal{M}$-points

Let X be a proper variety over a PF field (K, C) with a finite collection of closed subschemes $(D_\alpha)_{\alpha\in\mathcal{A}}$. Fix an integral model $\mathcal{X}$ of X over $B \subset C$ and set $\mathcal{D}_\alpha = \mathcal{D}_\alpha^c$. By choosing different subsets $\mathfrak{M} \subset \overline{\mathbb{N}}^{\mathcal{A}}$, we can construct many different pairs $(\mathcal{X}, \mathcal{M})$. We consider some choices, and afterwards we describe $\mathcal{M}$-points on projective space for these choices. We write $U = X \setminus \bigcup_{\alpha\in\mathcal{A}} D_\alpha$ and $\mathcal{U} = \mathcal{X} \setminus \bigcup_{\alpha\in\mathcal{A}} \mathcal{D}_\alpha$ for the complement of the boundary.

1. If $\mathfrak{M} = \{(0, \dots, 0)\}$, then the $\mathcal{M}$-points over B are the integral points on $\mathcal{U}$: $(\mathcal{X}, \mathcal{M})(B) = \mathcal{U}(B)$ and $(X, M)(K) = U(K)$.

   More generally, if $\mathcal{B} \subset \mathcal{A}$ and $\mathfrak{M} = \{\mathbf{m} \in \overline{\mathbb{N}}^{\mathcal{A}} \mid \mathbf{m}_\alpha = \mathbf{0} \text{ if } \alpha \in \mathcal{B}\}$, then the $\mathcal{M}$-points over B are the integral points on $\mathcal{X} \setminus \bigcup_{\alpha\in\mathcal{B}} \mathcal{D}_\alpha$: $(\mathcal{X}, \mathcal{M})(B) = (\mathcal{X} \setminus \bigcup_{\alpha\in\mathcal{B}} \mathcal{D}_\alpha)(B)$ and $(X, M)(K) = (X \setminus \bigcup_{\alpha\in\mathcal{B}} D_\alpha)(K)$.

2. If $\mathfrak{M} = \overline{\mathbb{N}}^{\mathcal{A}}$, then the set of $\mathcal{M}$-points is the entire set of rational points: $(\mathcal{X}, \mathcal{M})(B) = (X, M)(K) = X(K)$. If on the other hand $\mathfrak{M} = \mathbb{N}^{\mathcal{A}}$, then the set consists of only the points not contained in the boundary: $(\mathcal{X}, \mathcal{M})(B) = (X, M)(K) = U(K)$.

3. If $\mathfrak{M} = \{0, 1\}^{\mathcal{A}}$, then $(\mathcal{X}, \mathcal{M})(B)$ is the set of points on $\mathcal{X}$ over B that intersect all $\mathcal{D}_\alpha$ transversally. As we will see, we can think of these points as a sort of "squarefree" points. We again have $(X, M)(K) = U(K)$.

4. If

   $$\mathfrak{M} = \bigcup_{\alpha\in\mathcal{A}} \{\mathbf{w} \in \overline{\mathbb{N}}^{\mathcal{A}} \mid w_{\alpha'} = 0\, \forall \alpha' \neq \alpha\},$$

   then $(\mathcal{X}, \mathcal{M})(B)$ is the set of points on $\mathcal{X}$ over B which do not meet any intersection $\mathcal{D}_\alpha \cap \mathcal{D}_{\alpha'}$ for $\alpha, \alpha' \in \mathcal{A}$, $\alpha \neq \alpha'$, while $(X, M)(K)$ consists of the rational points not contained in any of of the intersections $D_\alpha \cap D_{\alpha'}$.

For the following examples we assume that the closed subschemes D_α and $\mathcal{D}_\alpha$ are prime Weil divisors, and $D_\alpha \neq D_{\alpha'}$ if $\alpha \neq \alpha'$. Consider a vector of multiplicities $\mathbf{m} = (m_\alpha)_{\alpha\in\mathcal{A}}$, where $m_\alpha \in \mathbb{N}^* \cup \{\infty\}$ and define the $\mathbb{Q}$-Weil divisor $\mathcal{D}_\mathbf{m} = \sum_{\alpha\in\mathcal{A}} \left(1 - \frac{1}{m_\alpha}\right) \mathcal{D}_\alpha$, where we set $\frac{1}{\infty} = 0$.

Definition 2.1.19. For $\mathcal{X}$ and $\mathcal{D}_\mathbf{m}$ as above, we define special points as follows.

- *Campana points on* $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ *over* B are the $\mathcal{M}$-points over B for the pair $(\mathcal{X}, \mathcal{M})$, where $\mathfrak{M}$ is the collection of $\mathbf{w} \in \overline{\mathbb{N}}^{\mathcal{A}}$ such that for all $\alpha \in \mathcal{A}$ we have
  1. $w_\alpha = 0$ if $m_\alpha = \infty$ and
  2. $w_\alpha = 0$ or $w_\alpha \geq m_\alpha$ if $m_\alpha \neq \infty$.
- *Weak Campana points on* $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ *over* B are the $\mathcal{M}$-points over B for $(X, \mathcal{M})$, where $\mathfrak{M}$ is the collection of all $\mathbf{w}$ such that
  1. $w_\alpha = 0$ if $m_\alpha = \infty$ and
  2. either $w_\alpha = 0$ for all $\alpha \in \mathcal{A}$ or
  $$\sum_{\substack{\alpha\in\mathcal{A}\\ m_\alpha\neq 1}} \frac{w_\alpha}{m_\alpha} \geq 1.$$
- *Strict Darmon points on* $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ *over* B are the $\mathcal{M}$-points where $\mathfrak{M}$ is the collection of $\mathbf{w} \in \mathbb{N}^{\mathcal{A}}$ for which $m_\alpha | w_\alpha$ for all $\alpha \in \mathcal{A}$. Here we use the convention that the only integer divisible by ∞ is 0. If we take the closure of $\mathfrak{M}$ in $\overline{\mathbb{N}}^{\mathcal{A}}$ then we obtain the *Darmon points on* $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ *over* B.

Note that if $m_\alpha = \infty$ for all $\alpha \in \mathcal{A}$, then all of the sets of $\mathcal{M}$-points in Definition 2.1.19 reduce to the set of integral points on $\mathcal{U}$.

Notation 2.1.20. We will sometimes refer to pairs $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ and pairs $\left(X, \sum_{\alpha\in\mathcal{A}} \left(1 - \frac{1}{m_\alpha}\right) D_\alpha\right)$ as *Campana pairs*, to distinguish them from the pairs considered in this thesis.

Note that all examples given in this section satisfy the property that $\mathfrak{M}$ is an open subset of $\overline{\mathbb{N}}^{\mathcal{A}}$, except for the set $\mathfrak{M}$ encoding the multiplicities for the Darmon points. This follows from the fact that $\mathbb{N} \subset \overline{\mathbb{N}}$ is an open subspace with the discrete topology and if $U \subset \overline{\mathbb{N}}$ contains ∞, then U is open if and only if it contains all integers greater than a fixed integer N_0. The set of multiplicities is also closed for the other examples except for the strict Darmon points, the integral points on $\mathcal{U}$ and the rational points on U.

If we additionally assume that X is smooth, $\mathcal{X}$ is flat over B and the D_α are Cartier divisors, then (weak) Campana points and Darmon points agree with their definition as given in [MNS24]. If $\sum_{\alpha\in\mathcal{A}} D_\alpha$ is furthermore a strict normal crossings divisor, X is geometrically integral and $\mathcal{X}$ is regular, then the (weak) Campana points agree with the definition given in [PSTVA21].

Remark 2.1.21. In [MNS24] strong Campana points and strong Darmon points are defined, of which the former were called Campana points in [NS24; Str22]. The set of strong Campana points and the set of strong Darmon points are generally not examples of sets of $\mathcal{M}$-points if the divisors $\mathcal{D}_\alpha$ are not geometrically integral, as those points are defined using the intersection multiplicities of the irreducible components of $\mathcal{D}_{\alpha,\mathcal{O}_v}$. However, if the D_α are geometrically integral then strong Campana points and strong Darmon points coincide with Campana points and Darmon points, respectively.

Remark 2.1.22. Consider a positive integer m and a prime Cartier divisor D on a smooth proper variety X extending to a prime Cartier divisor $\mathcal{D}$ on an integral model $\mathcal{X}$. Then the Campana points and weak Campana points on $(\mathcal{X}, (1-\frac{1}{m})\mathcal{D})$ coincide and agree with the weak Campana points as defined in [Str22], even if D is not geometrically irreducible.

$\mathcal{M}$-points on projective space

To further illustrate the examples given above, we fix $K = \mathbb{Q}$, $B = \operatorname{Spec}\mathbb{Z}$ and $X = \mathbb{P}^n_{\mathbb{Q}}$ with integral model $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$. Fix $\mathcal{A} = \{0, \dots, n\}$. For $i \in \{1, \dots, n\}$, we define D_i to be the coordinate hyperplane $\{x_i = 0\}$ and let $\mathcal{D}_i$ be the Zariski closure of D_i in $\mathcal{X}$. Then for a point $P = (a_0 : \dots : a_n)$ with $a_i \in \mathbb{Z}$ for all $i \in \{1, \dots, n\}$ and $\gcd(a_0, \dots, a_n) = 1$, the identity $n_p(D_i, P) = v_p(a_i)$ holds for every prime number p. In particular, given a set $\mathfrak{M} \subset \overline{\mathbb{N}}^{\{0,\dots,n\}} = \overline{\mathbb{N}}^{n+1}$, we see that $(\mathbb{P}^n, \mathcal{M})(\mathbb{Z})$ is equal to

$$\{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid (v_p(a_0), \dots, v_p(a_n)) \in \mathfrak{M} \text{ for all prime numbers } p\}.$$

So in particular:

- If $\mathfrak{M} = \{0,1\}^{n+1}$, then
$$(\mathbb{P}^n, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid a_0, \dots, a_n \text{ squarefree}\}.$$
- If $\mathfrak{M} = \bigcup_{i=0}^n \{\mathbf{w} \in \overline{\mathbb{N}}^{n+1} \mid w_j = 0 \, \forall j \neq i\}$, then
$$(\mathbb{P}^n, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid \gcd(a_i, a_j) = 1 \, \forall i, j, \, i \neq j\}.$$
- The set of Campana points for the multiplicities $m_0, \dots, m_n$ is
$$(\mathbb{P}^n, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid a_i \text{ is } m_i\text{-full}\}.$$
Here we recall that for an integer $m \geq 1$, we say that an integer n is m-full if $p|n$ implies $p^m|n$ for every prime number p. Furthermore, we define -1 and 1 to be the only ∞-full numbers. Additionally, we will cal 2-full numbers squareful.
- The set of Darmon points for the multiplicities $m_0, \dots, m_n$ is
$$(\mathbb{P}^n, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid \ |a_i| \text{ is an } m_i\text{-th power}\}.$$
- If $\mathfrak{M} = \{\mathbf{w} \in \overline{\mathbb{N}}^{n+1} \mid w_i \leq w_j \text{ if } i \leq j\}$, then
$$(\mathbb{P}^n, \mathcal{M})(\mathbb{Z}) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n(\mathbb{Z}) \mid \ a_i \text{ divides } a_j \text{ if } i \leq j\}.$$

Remark 2.1.23. The above descriptions easily generalize to the case where $B = \operatorname{Spec} R$ for a principal ideal domain R. If on the other hand R is not a principal ideal domain, then more care is needed since then it is not possible to write every rational point P as $P = (a_0 : \cdots : a_n)$ such that there is an equality $(a_0, \ldots, a_n) = R$ as ideals. For example, if $R = \mathbb{Z}[\sqrt{-5}]$, then the rational point $(2 : 1 + \sqrt{-5}) \in \mathbb{P}^1(\mathbb{Q}(\sqrt{-5}))$ cannot be written in such a form.

For smooth split toric varieties with D_α the torus-invariant prime divisors, we will see that we have a very similar concrete description for $\mathcal{M}$-points, see Remark 3.2.5.

2.1.5 Equivalence of pairs

In the remainder of this chapter and all of Chapter 3, X is a variety over a PF field (K, C), (X, M) is a pair and $B \subset C$ is an open subscheme, unless specified otherwise. It is clear from the definition of $\mathcal{M}$-points that some pairs $(\mathcal{X}, \mathcal{M})$, $(\mathcal{X}, \mathcal{M}')$ have the same set of points or one is contained in the other for geometric reasons. Therefore we introduce the following definition.

Definition 2.1.24. Let (X, M) and (X, M') be pairs with integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}', \mathcal{M}')$ over B. Then we say that $(\mathcal{X}, \mathcal{M})$ is *equivalent* to $(\mathcal{X}', \mathcal{M}')$ if

$$(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) = (\mathcal{X}', \mathcal{M}')(\mathcal{O}_v) \text{ as subsets of } X(K_v)$$

for all places $v \in B$. If $\mathcal{X} = \mathcal{X}'$ we simply say that $\mathcal{M}$ is equivalent to $\mathcal{M}'$. Similarly we write $(\mathcal{X}, \mathcal{M}) \subset (\mathcal{X}', \mathcal{M}')$ if

$$(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) \subset (\mathcal{X}', \mathcal{M}')(\mathcal{O}_v) \text{ as subsets of } X(K_v)$$

for all places $v \in \Omega_K$. If $\mathcal{X} = \mathcal{X}'$ we simply write $\mathcal{M} \subset \mathcal{M}'$.

If there exists a nonempty open subset $B \subset C$ and equivalent integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}', \mathcal{M}')$ over B, then we say that M is equivalent to M'. Similarly if there exists a nonempty open subset $B \subset C$ and integral models $(\mathcal{X}, \mathcal{M}) \subset (\mathcal{X}', \mathcal{M}')$, then we write $M \subset M'$.

The next proposition implies that any two integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}, \mathcal{M}')$ of (X, M) over B become equivalent over some nonempty open subset $B' \subset B$.

Proposition 2.1.25. *Let (X, M) be a pair with integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}', \mathcal{M}')$ over B. Then there is a nonempty open subset $B' \subset B$ such that there is an isomorphism $f\colon \mathcal{X}_{B'} \to \mathcal{X}'_{B'}$ such that $\mathcal{D}_{\alpha,B'}$ maps isomorphically to $\mathcal{D}'_{\alpha,B'}$ for all $\alpha \in \mathcal{A}$. In particular, $(\mathcal{X}, \mathcal{M})_{B'}$ is equivalent to $(\mathcal{X}', \mathcal{M}')_{B'}$.*

Proof. As we can restrict to an open subset of B, we can without loss of generality assume that $\mathcal{X}$ and $\mathcal{X}'$ are flat over B. The proof works via spreading out, and is analogous to [Poo17, Theorem 3.2.1(iii)]. Here we use the fact that $\mathcal{X}$ and $\mathcal{X}'$ are finitely presented over B and therefore $\mathcal{D}_\alpha$ and $\mathcal{D}'_\alpha$ are as well. [Poo17, Theorem 3.2.1(iii)] implies that the identity id_X lifts to a morphism $f\colon \mathcal{X}_{B'} \to \mathcal{X}'_{B'}$ and a morphism $g\colon \mathcal{X}'_{B'} \to \mathcal{X}_{B'}$. By the same reasoning, we can take $B' \subset B$ a small enough open such that, for all $\alpha \in \mathcal{A}$, we have $f(\mathcal{D}_{\alpha,B'}) \subset \mathcal{D}'_{\alpha,B'}$ and $g(\mathcal{D}'_{\alpha,B'}) \subset \mathcal{D}_{\alpha,B'}$ as schemes. As $g \circ f\colon \mathcal{X}_{B'} \to \mathcal{X}_{B'}$ and $f \circ g\colon \mathcal{X}'_{B'} \to \mathcal{X}'_{B'}$ restrict to the identity on X_K, it follows from [EGA4, Théorème 8.10.5(i)] that there exists a nonempty open subset $B'' \subset B'$ such that $f_{B''}$ and $g_{B''}$ are isomorphisms and are inverses of each other and therefore also identify $\mathcal{D}_{\alpha,B''}$ and $\mathcal{D}'_{\alpha,B''}$ under the isomorphism. □

It can sometimes be convenient to remove the elements in $\mathfrak{M}$ which correspond to empty intersections of the boundary components D_α in X, which is why we make the following definition.

Definition 2.1.26. Let (X, M) be a pair. Consider the subset $\mathfrak{M}_{\mathrm{red}} \subset \mathfrak{M}$ defined by

$$\mathfrak{M}_{\mathrm{red}} = \{\mathbf{m} \in \mathfrak{M} \mid \cap_{m_\alpha \neq 0} D_\alpha \neq \emptyset\}.$$

We write $M_{\mathrm{red}} = ((D_\alpha)_{\alpha\in\mathcal{A}}, \mathfrak{M}_{\mathrm{red}})$. If $\mathfrak{M} = \mathfrak{M}_{\mathrm{red}}$, then we call $\mathfrak{M}$ *reduced.*

Proposition 2.1.27. *If (X, M) is a pair, then M and M_{red} are equivalent.*

Proof. Let $\mathcal{X}$ be an integral model of X over an open subset $B \subset C$. Then for all subsets $V \subset \mathcal{A}$ with $\cap_{\alpha\in V} D_\alpha = \emptyset$, there exists a nonempty open subset $B_V \subset B$ such that $\cap_{\alpha\in V} \mathcal{D}^c_{\alpha,B_V} = \emptyset$. The intersection $B' = \cap B_V$ over all such V is a nonempty open subset of B, since $\mathcal{A}$ is finite. In particular, we see that for any integral model $(\mathcal{X}, \mathcal{M})$ of (X, M) over B, $(\mathcal{X}, \mathcal{M})_{B'}$ is equivalent to $(\mathcal{X}, \mathcal{M}_{\mathrm{red}})_{B'}$. □

Example 2.1.28. To illustrate the difference between M and M_{red}, we take $X = \mathbb{P}^1_{\mathbb{Q}}$ with integral model $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and consider the disjoint divisors $D_1 = \{X_0 = 0\}$ and $D_2 = \{X_0 - 2X_1 = 0\}$ on $\mathbb{P}^1_{\mathbb{Q}}$ defined by the homogeneous ideals (X_0) and $(X_0 - 2X_1)$. Then $\mathcal{D}^c_1$ and $\mathcal{D}^c_2$ are the divisors defined by the same ideals and $\mathcal{D}^c_1 \cap \mathcal{D}^c_2$ is the subscheme defined by the homogeneous ideal $(2X_1, X_0)$. Note that $(2X_1, X_0)$ and $(2, X_0)$ define the same subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ since X_0 and X_1 cannot vanish simultaneously. We define

$$M = ((D_1, D_2), \overline{\mathbb{N}}^2),$$

so

$$M_{\mathrm{red}} = ((D_1, D_2), \{0\} \times \overline{\mathbb{N}} \cup \overline{\mathbb{N}} \times \{0\}).$$

Then $(\mathcal{X}, \mathcal{M}^c)(\mathbb{Z}) = \mathbb{P}^1(\mathbb{Q})$, while

$$(\mathcal{X}, (\mathcal{M}_{\mathrm{red}})^c)(\mathbb{Z}) = \{(x : y) \in \mathbb{P}^1(\mathbb{Q}) \mid x, y \in \mathbb{Z},\ \gcd(x, y) = 1,\ x \text{ odd}\}$$

since $\mathcal{D}^c_2 \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_2 = \mathcal{D}^c_1 \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_2 = \{X_0 = 0\} \subset \mathbb{P}^1_{\mathbb{F}_2}$, where $\mathbb{F}_2$ is the field with two elements. However, if we invert 2, then the two pairs become equivalent: $(\mathcal{X}, \mathcal{M}^c)(\mathbb{Z}[\frac{1}{2}]) = (\mathcal{X}, (\mathcal{M}_{\mathrm{red}})^c)(\mathbb{Z}[\frac{1}{2}]) = \mathbb{P}^1(\mathbb{Q})$.

Example 2.1.29. Let X be a curve with disjoint divisors D_α and multiplicities $m_\alpha \in \mathbb{N}^* \cup \{\infty\}$ for $\alpha \in \mathcal{A}$, and consider the pairs (X, M) and (X, M') associated to Campana points and weak Campana points on $(X, D_{\mathbf{m}})$ as in Definition 2.1.19. Then by Proposition 2.1.27 the pairs (X, M) and (X, M') are equivalent as $M'_{\mathrm{red}} = M_{\mathrm{red}}$. In particular, for every choice of integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}, \mathcal{M}')$ there exists an open subset $B \subset C$ such that $(\mathcal{X}, \mathcal{M})(B') = (\mathcal{X}, \mathcal{M}')(B')$ whenever $B' \subset B$ is an open subset.

2.1.6 Inverse image of a pair

If we have a pair (X, M) and a morphism $f\colon Y \to X$, we often want to to pull back the structure on X to Y to get a pair $(Y, f^{-1}M)$ such that, given a lift $f\colon \mathcal{Y} \to \mathcal{X}$ between integral models, there is an equality

$$f^{-1}((\mathcal{X}, \mathcal{M})(B)) = (\mathcal{Y}, f^{-1}\mathcal{M})(B)$$

of subsets of $\mathcal{Y}(B)$.

Definition 2.1.30. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over a scheme B and let $(\mathcal{X}, \mathcal{M})$ be a pair over B. We define the inverse image of $(\mathcal{X}, \mathcal{M})$ under f to be the pair $(\mathcal{Y}, f^{-1}\mathcal{M})$, where

$$f^{-1}\mathcal{M} = ((f^{-1}\mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \mathfrak{M}),$$

and $f^{-1}\mathcal{D}_\alpha := \mathcal{D}_\alpha \times_{\mathcal{X}} \mathcal{Y}$.

Proposition 2.1.31. *Let (K, C) be a PF field, let $(\mathcal{X}, \mathcal{M})$ be a pair over B and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over an open subscheme $B \subset C$, where $\mathcal{Y}$ is an integral model over B of a variety Y over K. Then for all closed points $v \in B$,*

$$(\mathcal{Y}, f^{-1}\mathcal{M})(\mathcal{O}_v) = \{P \in Y(K_v) \mid f(P) \in (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)\},$$

and therefore

$$(\mathcal{Y}, f^{-1}\mathcal{M})(B) = \{P \in Y(K) \mid f(P) \in (\mathcal{X}, \mathcal{M})(B)\}.$$

Proof. Let $v \in \Omega_K$, let $\mathcal{P}_v \in \mathcal{Y}(\mathcal{O}_v)$ and let $\alpha \in \mathcal{A}$. Since every square in the diagram

$$\begin{array}{ccccc}
\mathcal{P}_v \cap f^{-1}\mathcal{D}_\alpha & \longrightarrow & f^{-1}\mathcal{D}_\alpha & \longrightarrow & \mathcal{D}_\alpha \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}\mathcal{O}_v & \xrightarrow{\mathcal{P}_v} & \mathcal{Y} & \xrightarrow{f} & \mathcal{X}
\end{array}$$

is Cartesian, the fiber product of $f \circ \mathcal{P}_v$ and $\mathcal{D}_\alpha \to \mathcal{X}$ is the the same as the fiber product of $\mathcal{P}_v$ and $f^{-1}\mathcal{D}_\alpha \to \mathcal{X}$. Therefore, $n_v(\mathcal{D}_\alpha, f \circ \mathcal{P}_v) = n_v(f^{-1}\mathcal{D}_\alpha, \mathcal{P}_v)$, and thus the desired equalities hold. $\square$

Note that taking the induced integral model does not need to commute with taking the inverse image since we can have

$$(\mathcal{Y}, (f^{-1}\mathcal{M})^c)(B) \subsetneq (\mathcal{Y}, f^{-1}(\mathcal{M}^c))(B),$$

as the next example shows.

Example 2.1.32. Let $K = k(t)$, for k a field, and let $X = Y = \mathbb{P}^1_K$, $f = \mathrm{id}_X$ and $D = \{(0:1)\} \in X$. Choose the integral model $\mathcal{X} = \mathbb{P}^1_k \times_k \mathbb{A}^1_k$ of X over $\mathbb{A}^1_k$, let $P = ((0:1), 0) \in \mathcal{X}$, and let $\mathcal{Y} = \mathrm{Bl}_P \mathcal{X}$ be the blowup of $\mathcal{X}$ in P. Then $\mathcal{Y}$ is an integral model of Y and f lifts to the blowup morphism $f\colon \mathcal{Y} \to \mathcal{X}$. Furthermore $(f^{-1}\mathcal{D})^c$ is the strict transform of $\mathcal{D}^c$, which is strictly contained in the inverse image: $f^{-1}(\mathcal{D}^c) = (f^{-1}\mathcal{D})^c \cup f^{-1}(P)$. If we restrict the models to the open subset $B = \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$, this discrepancy disappears and we find that $\mathcal{X}_B \cong \mathcal{Y}_B$.

Note that if $f\colon Y \to X$ is a dominant map of integral varieties and the closed subschemes D_α are Cartier divisors, the schemes $f^{-1}D_\alpha$ are Cartier divisors [Stacks, Tag 02OO], but usually not prime divisors. It can be convenient to have a pair equivalent to $(Y, f^{-1}M)$ where the chosen closed subschemes are prime divisors, so we introduce the following notion.

Definition 2.1.33. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over a scheme B and let $(\mathcal{X}, \mathcal{M})$ be a pair over B. Assume furthermore that $f^{-1}\mathcal{D}_\alpha$ is a sum of prime Cartier divisors for all $\alpha \in \mathcal{A}$. Then we define the *pullback of* $(\mathcal{X}, \mathcal{M})$ *under* f to be the pair $(\mathcal{Y}, f^*\mathcal{M})$, where $f^*\mathcal{M} = (\{\widetilde{D}_\beta\}_{\beta\in\mathcal{B}}, f^*\mathfrak{M})$. The $\widetilde{\mathcal{D}}_\beta$ are the prime divisors on $\mathcal{Y}$ contained in $f^{-1}\mathcal{D}_\alpha$ for some $\alpha \in \mathcal{A}$, without repetitions. We define

$$f^*\mathfrak{M} = \left\{ w' \in \overline{\mathbb{N}}^{\mathcal{B}} \;\middle|\; \left(\sum_{\beta\in\mathcal{B}} c_{\alpha,\beta} w'_\beta \right)_{\alpha\in\mathcal{A}} \in \mathfrak{M} \right\},$$

where the $c_{\alpha,\beta}$ are given by

$$f^{-1}\mathcal{D}_\alpha = \sum_{\beta\in\mathcal{B}} c_{\alpha,\beta} \widetilde{\mathcal{D}}_\beta.$$

Note that this definition is unique up to changing the indexing on the divisors $\widetilde{\mathcal{D}}_\beta$. As a consequence of Proposition 2.1.11, $(\mathcal{Y}, f^*\mathcal{M})$ is equivalent to $(\mathcal{Y}, f^{-1}\mathcal{M})$.

2.2 Adelic M-points and M-approximation

2.2.1 Adelic M-points

In this section we introduce adelic M-points and integral adelic $\mathcal{M}$-points. Recall from Section 1.3.4 that the adele ring of a PF field (K, C) prime to a finite set of places $T \subset \Omega_K$ is the restricted product

$$\mathbf{A}_K^T = \prod_{v\in\Omega_K\setminus T} (K_v, \mathcal{O}_v),$$

which is given the structure of a topological K-algebra as in Definition 1.3.7. Using the restricted product, we generalize the notion of adelic points on a variety X to adelic M-points on a pair (X, M).

Definition 2.2.1. Let (K, C) be a PF field, let $T \subset \Omega_K$ be a finite sets of places, and let $B \subset C$ be an open subscheme. Let (X, M) be a pair over (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over B. We define the space of *integral adelic* $\mathcal{M}$*-points over* B *prime to* T to be the product

$$(\mathcal{X}, \mathcal{M})(\mathbf{A}_B^T) = \prod_{v\in\Omega_K\setminus T} (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$$

with the product topology. The space of *adelic* M*-points over* B *prime to* T is defined as the restricted product

$$(X, M)(\mathbf{A}_K^T) = \prod_{v\in\Omega_K\setminus T} ((X, M)(K_v), (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)).$$

In Section 2.3 we will generalize these notions further in order to compare them to the adelic points considered in [MNS24]. While the space of integral adelic $\mathcal{M}$-points depends on the choice of an integral model, even as a set, the space of adelic M-points does not depend on such a choice.

Proposition 2.2.2. *If (X, M) is a pair over (K, C) with integral models $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}', \mathcal{M}')$ over $B \subset C$, and $T \subset \Omega_K$ is a finite set of places, then there is a canonical homeomorphism*

$$\prod_{v\in\Omega_K\setminus T}((X,M)(K_v),(\mathcal{X},\mathcal{M})(\mathcal{O}_v)) \to \prod_{v\in\Omega_K\setminus T}((X,M)(K_v),(\mathcal{X}',\mathcal{M}')(\mathcal{O}_v)).$$

Proof. By Proposition 2.1.25 there exists a nonempty open $B' \subset B$ over which $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{X}', \mathcal{M}')$ are equivalent. Denote by S' the set of places in $B \setminus (B' \cup T)$. By the properties of the restricted product as recalled in Section 1.3.5,

$$\begin{aligned}
&\prod_{v\in\Omega_K\setminus T}((X,M)(K_v),(\mathcal{X},\mathcal{M})(\mathcal{O}_v)) \\
&\cong \prod_{v\in S'}(X,M)(K_v) \times \prod_{v\in\Omega_K\setminus(T\cup S')}((X,M)(K_v),(\mathcal{X},\mathcal{M})(\mathcal{O}_v)) \\
&\cong \prod_{v\in\Omega_K\setminus T}((X,M)(K_v),(\mathcal{X}',\mathcal{M}')(\mathcal{O}_v))
\end{aligned}$$

□

In particular, $(X, M)(\mathbf{A}_K^T)$ is well-defined for any pair (X, M), because every pair has an integral model.

Example 2.2.3. If $\mathfrak{M} = \mathbb{N}^{\mathcal{A}}$, then

$$(\mathcal{X},\mathcal{M})(\mathbf{A}_B^T) = (X,M)(\mathbf{A}_K^T) = \prod_{v\in\Omega_K\setminus T} U(K_v),$$

where $U = X \setminus \bigcup_{\alpha\in\mathcal{A}} D_\alpha$.

Example 2.2.4. If $\mathfrak{M} = \{(0, \dots, 0)\}$, then the space of integral adelic points on $\mathcal{U}$ (defined in [LS16, page 2] as S-adelic points) is the space of integral adelic $\mathcal{M}$-points on $\mathcal{X}$:

$$\mathcal{U}(\mathbf{A}_B^T) := (\mathcal{X},\mathcal{M})(\mathbf{A}_B^T) = \prod_{v\in\Omega_K\setminus S\cup T}\mathcal{U}(\mathcal{O}_v) \times \prod_{v\in S\setminus T} U(K_v),$$

where we write $S = \Omega_K \setminus B$, $U = X \setminus \bigcup_{\alpha\in\mathcal{A}} D_\alpha$ and $\mathcal{U} = \mathcal{X} \setminus \bigcup_{\alpha\in\mathcal{A}} \mathcal{D}_\alpha$. The space of adelic points on U as in [LS16, page 2] is the space of adelic M-points:

$$U(\mathbf{A}_K^T) := (X,M)(\mathbf{A}_K^T) = \prod_{v\in\Omega_K\setminus S\cup T}(U(K_v),\mathcal{U}(\mathcal{O}_v)) \times \prod_{v\in S\setminus T} U(K_v).$$

Integral adelic $\mathcal{M}$-points also generalize the notion of adelic Campana points given in [PSTVA21; NS24]: if $\mathcal{M}$ encodes the Campana condition for a divisor $\mathcal{D}_{\mathbf{m}}$ as in Definition 2.1.19, then we have an equality of topological spaces

$$(\mathcal{X},\mathcal{M})(\mathbf{A}_B^T) = (\mathcal{X},\mathcal{D}_{\mathbf{m}})(\mathbf{A}_K^T),$$

where the right hand side is defined as in [PSTVA21, Section 3.2] and [NS24, Definition 2.4].

Given inclusions $(X, M) \subset (X, M')$, the natural injection $(X, M)(\mathbf{A}_K^T) \hookrightarrow (X, M')(\mathbf{A}_K^T)$ is continuous but it need not be a topological embedding. This is because generally the restricted product topology is strictly finer than the subspace topology on the product. This map does have dense image, as the next proposition shows.

Proposition 2.2.5. *Let $(X, M) \subset (X, M')$ be pairs over (K, C) and let $(\mathcal{X}, \mathcal{M})$ be an integral model of the former pair over an open subscheme $B \subset C$, and let $T \subset \Omega_K$ be a finite set of places. Then:*

1. *The natural inclusion $(X, M)(\mathbf{A}_K^T) \hookrightarrow (X, M')(\mathbf{A}_K^T)$ is continuous. Furthermore, it has dense image if for all $v \in \Omega_K \setminus T$, the subset $(X, M)(K_v) \subset (X, M')(K_v)$ is dense and $(X, M)(\mathbf{A}_K^T) \neq \emptyset$. The former assumption is automatic if X is smooth and none of the D_α contain irreducible components of X.*

2. *The natural inclusion $(\mathcal{X}, \mathcal{M})(\mathbf{A}_B^T) \hookrightarrow (X, M)(\mathbf{A}_K^T)$ is a topological embedding. Furthermore, if $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) \subset (X, M)(K_v)$ is open for all $v \in B$, then the inclusion is an open embedding.*

Proof. These statements follow from general properties of restricted products and are a special case of Proposition 1.3.6. If X is connected and none of the D_α contain irreducible components of X, then $(X, M_{\text{fin}})(K_v) \subset X(K_v)$ is dense by Proposition 2.2.6, proved below. □

Proposition 2.2.6. *Let X be a connected smooth variety over a PF field (K, C) and let $U \subset X$ be a nonempty open subset. Then $U(K_v) \subset X(K_v)$ is dense for all places $v \in \Omega_K$. Therefore any nonempty analytic open $V \subset X(K_v)$ is Zariski dense.*

Proof. Consider the complement $Z = X \setminus U$. Since X is smooth, [Con12, Lemma 5.3] allows us to reduce to the case when $X = \mathbb{A}_K^n$ and Z is contained in the zero locus of a single nonzero polynomial P. If $X(K_v) = K_v^n$ contained a nonempty open V such that $V \subset Z(K_v)$, then P vanishes identically on V. For any point $u \in V$ and a line $L \subset K_v^n$ through x, P vanishes identically on L since $L \cap V$ is infinite and any nonzero univariate polynomial only has finitely many zeroes. Since for any two points in K_v^n there exists a line through these points, this implies that P vanishes on K_v^n and therefore it is the zero polynomial, which is a contradiction. Thus $Z(K_v)$ contains no nonempty open sets in $X(K_v)$, so $U(K_v) \subset X(K_v)$ is dense.

Suppose X is connected and $V \subset X(K_v)$ is an analytic open. By the previous part, V cannot lie in a proper closed subscheme of X, and therefore V is Zariski dense in X. □

Example 2.2.7. If $M \subset M'$, then Proposition 2.2.5 implies that the natural inclusion $(X, M)(\mathbf{A}_K^T) \to (X, M')(\mathbf{A}_K^T)$ has dense image. The analogous statement for integral adelic $\mathcal{M}$-points is not always true, however. Consider $\mathcal{X} = \mathbb{P}_\mathbb{Z}^1$, $M = ((1:0), \{0\})$ and $M' = ((1:0), \{0, \infty\})$. Then the set of points in $(\mathcal{X}, \mathcal{M}^c)(\mathbb{Z}_p) = \{(a:1) \mid a \in \mathbb{Z}_p\}$ is a closed subset of $\mathbb{P}_\mathbb{Z}^1(\mathbb{Z}_p)$ which is strictly smaller than $(\mathcal{X}, \mathcal{M}'^c)(\mathbb{Z}_p) = \{(a:1) \mid a \in \mathbb{Z}_p\} \cup \{(1:0)\}$.

2.2.2 M-approximation

We now generalize the notion of strong approximation to $\mathcal{M}$-points.

Definition 2.2.8. Let (K, C) be a PF-field, let $T \subset \Omega_K$ be a finite set of places, let (X, M) be a pair over (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over $B \subset C$. Then we say that $(\mathcal{X}, \mathcal{M})$ satisfies *integral $\mathcal{M}$-approximation off T* if

$$(\mathcal{X}, \mathcal{M})(B) \hookrightarrow (\mathcal{X}, \mathcal{M})(\mathbf{A}_B^T)$$

has dense image, and we say that (X, M) satisfies *M-approximation off T* if

$$(X, M)(K) \hookrightarrow (X, M)(\mathbf{A}_K^T)$$

has dense image. We say that (X, M) satisfies M-approximation if it satisfies M-approximation off $T = \emptyset$.

Note that these notions only depend on the equivalence classes of $(\mathcal{X}, \mathcal{M})$ and (X, M).

If we take (X, M) to encode the integrality condition of an open subset $U \subset X$, then we recover strong approximation as in [Poo17, §2.6.4.5].

Definition 2.2.9. The open subset $U \subset X$ satisfies *strong approximation off T* if (X, M) satisfies M-approximation off T with $\mathfrak{M} = \{(0, \dots, 0)\}$, where $U = X \setminus \bigcup_{\alpha \in \mathcal{A}} D_\alpha$. Explicitly this says that

$$U(K) \hookrightarrow U(\mathbf{A}_K^T)$$

has dense image. If $\mathcal{M} = 0$, then we also say that $X = U$ satisfies *weak approximation off T*.

Example 2.2.10. If $\mathcal{M}$ is the Campana condition for $\mathcal{D}_{\mathbf{m}}$ as defined in Definition 2.1.19, then integral $\mathcal{M}$-approximation for $(\mathcal{X}, \mathcal{M})$ coincides with weak Campana approximation for $(\mathcal{X}, \mathcal{D}_{\mathbf{m}})$ as studied in [NS24].

Now we will relate the different notions of approximation to each other. The next proposition shows that integral $\mathcal{M}$-approximation is equivalent to integral $\mathcal{M}_{\text{fin}}$-approximation, provided that the $\mathcal{M}$-points in the boundary lie in the closure of the set of $\mathcal{M}_{\text{fin}}$-points.

Proposition 2.2.11. *Let $T \subset \Omega_K$ be a finite set of places, let (X, M) be a pair over (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over $B \subset C$.*

1. *If (X, M) satisfies M-approximation off T, then (X, M_{fin}) satisfies M_{fin}-approximation off T. If for every place $v \in \Omega_K \setminus T$, the subset $(X, M_{\text{fin}})(K_v) \subset (X, M)(K_v)$ is dense, then the converse also holds.*

2. *If $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T, then $(\mathcal{X}, \mathcal{M}_{\text{fin}})$ satisfies $\mathcal{M}_{\text{fin}}$-approximation off T. If for every place $v \in \Omega_K \setminus T$, the subset $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathcal{O}_v) \subset (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ is dense, then the converse also holds.*

Proof. The first statement follows from the fact that for any place $v \in \Omega_K \setminus T$, $(X, M_{\text{fin}})(K_v)$ is open in $(X, M)(K_v)$. The converse statement follows from the fact that $(X, M_{\text{fin}})(\mathbf{A}_K^T) \to (X, M)(\mathbf{A}_K^T)$ has dense image by Proposition 2.2.5. Now we prove the statements for integral $\mathcal{M}$-approximation. Assume that $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T. Then in particular, for any place $v \in \Omega_K \setminus T$, $(\mathcal{X}, \mathcal{M}_{\text{fin}})(B)$ is dense in $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathcal{O}_v) \times (\mathcal{X}, \mathcal{M})(\mathbf{A}_B^{T \sqcup \{v\}})$, since $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathcal{O}_v)$ is open in $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ by Proposition 2.1.2. Therefore every open $\prod_{v \in \Omega_K \setminus T} U_v \subset (\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathbf{A}_B^{T \sqcup \{v\}})$ contains an element in $(\mathcal{X}, \mathcal{M}_{\text{fin}})(B)$, so $(\mathcal{X}, \mathcal{M}_{\text{fin}})$ satisfies integral $\mathcal{M}_{\text{fin}}$-approximation off T. If $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathcal{O}_v)$ is a dense subset of $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ for all $v \in \Omega_K \setminus T$, then the inclusion map $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathbf{A}_B^T) \to (\mathcal{X}, \mathcal{M})(\mathbf{A}_B^T)$ has dense image. Therefore, if $(\mathcal{X}, \mathcal{M}_{\text{fin}})$ satisfies integral $\mathcal{M}_{\text{fin}}$-approximation off T, then $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T. ☐

Using the Proposition 2.2.11, we are able to relate integral $\mathcal{M}$-approximation and M-approximation.

Proposition 2.2.12. *Let (X, M) be a pair over (K, C) with an integral model $(\mathcal{X}, \mathcal{M})$ over $B \subset C$. Let $T \subset \Omega_K \setminus B$ be a set of places disjoint from the points of B. If (X, M) satisfies M-approximation off T, then $(\mathcal{X}, \mathcal{M}_{\text{fin}})$ satisfies integral $\mathcal{M}_{\text{fin}}$-approximation off T.*

Proof. By Proposition 2.2.11, (X, M_{fin}) satisfies M_{fin}-approximation off T. By Proposition 2.2.5, the inclusion $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathbf{A}_B^T) \hookrightarrow (X, M_{\text{fin}})(\mathbf{A}_K^T)$ is an open embedding, and $(X, M_{\text{fin}})(K) \cap (\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathbf{A}_B^T) = (\mathcal{X}, \mathcal{M}_{\text{fin}})(B)$. Since $(X, M_{\text{fin}})(K)$ is dense in $(X, M_{\text{fin}})(\mathbf{A}_K^T)$, $(\mathcal{X}, \mathcal{M}_{\text{fin}})(B)$ is dense in $(\mathcal{X}, \mathcal{M}_{\text{fin}})(\mathbf{A}_B^T)$ as the intersection of a dense set with an open subset is dense in the open subset. ☐

The following proposition is an analog of [MNS24, Proposition 3.22] (and a generalization of [NS24, Lemma 2.8]). It is a partial converse to Proposition 2.2.12 and that states if integral $\mathcal{M}$-approximation on $(\mathcal{X}, \mathcal{M})$ is preserved under restricting the base B, then (X, M) satisfies M-approximation.

Proposition 2.2.13. *Let (X, M) be a pair over (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over $B \subset C$ and let $T \subset \Omega_K$ be a finite set of places. If for every nonempty open subscheme $B' \subset B$, $(\mathcal{X}, \mathcal{M})_{B'}$ satisfies integral $\mathcal{M}_{B'}$-approximation off T, then (X, M) satisfies M-approximation off T. If furthermore, $\mathfrak{M} \subset \overline{\mathbb{N}}^{\mathcal{A}}$ is open, then the converse also holds.*

Proof. We first assume that $(\mathcal{X}, \mathcal{M})_{B'}$ satisfies integral $\mathcal{M}_{B'}$-approximation off T for all $B' \subset B$. Note that every $P \in (X, M)(\mathbf{A}_K^T)$ lies in the subspace $(\mathcal{X}, \mathcal{M})_{B'}(\mathbf{A}_{B'}^T)$ for some $B' \subset B$. Thus since every open neighbourhood of P in $(\mathcal{X}, \mathcal{M})_{B'}(\mathbf{A}_{B'}^T)$ has nonempty intersection with $(\mathcal{X}, \mathcal{M})_{B'}(B')$, the same holds for any open neighbourhood of P in $(X, M)(\mathbf{A}_K^T)$. Hence (X, M) satisfies M-approximation off T.

In the other direction, note that if $\mathfrak{M} \subset \overline{\mathbb{N}}^{\mathcal{A}}$ is open, then $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) \subset (X, M)(K_v)$ is open by Proposition 2.1.2. Therefore, by Proposition 2.2.5, $(\mathcal{X}, \mathcal{M})_{B'}(\mathbf{A}_{B'}^T)$ is open in $(X, M)(\mathbf{A}_K^T)$ so $(X, M)(K) \cap (\mathcal{X}, \mathcal{M})_{B'}(\mathbf{A}_{B'}^T) = (\mathcal{X}, \mathcal{M})_{B'}(B')$ is dense in $(\mathcal{X}, \mathcal{M})_{B'}(\mathbf{A}_{B'}^T)$. ☐

Using the concept of an inverse image of a pair introduced in Definition 2.1.30, we can use birational morphisms, such as a resolution of singularities, to understand M-approximation.

Proposition 2.2.14. *Let (K, C) be a PF field, let $T \subset \Omega_K$ be a finite set of places and let $f : Y \to X$ be a birational morphism of proper integral K-varieties, where Y is smooth over K. Suppose that the induced map $Y(K_v) \to X(K_v)$ is surjective for all places $v \in \Omega_K \setminus T$. Then any pair (X, M) satisfies M-approximation off T if and only if $(Y, f^{-1}M)$ satisfies $f^{-1}M$-approximation off T.*

If $B \subset C$ is an open subscheme and $f : Y \to X$ spreads out to a morphism $f : \mathcal{Y} \to \mathcal{X}$ of integral models, then any pair $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T if and only if $(\mathcal{Y}, f^{-1}\mathcal{M})$ satisfies integral $f^{-1}\mathcal{M}$-approximation off T.

Proof. By the assumption, it follows that the induced map $(Y, f^{-1}M)(\mathbf{A}_K^T) \to (X, M)(\mathbf{A}_K^T)$ is surjective by Proposition 2.1.31. Therefore, if $(Y, f^{-1}M)$ satisfies $f^{-1}M$-approximation, then (X, M) satisfies M-approximation. In the other direction, if $V \subset Y$ is the locus over which f is an isomorphism, then $f^{-1}((X, M)(K)) \cap V(K)$ is dense in $(Y, f^{-1}M)(\mathbf{A}_K^T) \cap \prod_{v \in \Omega_K \setminus T} V(K_v)$. Since Y is smooth, $V(K_v)$ is dense in $Y(K_v)$ by Proposition 2.2.6, so $f^{-1}((X, M)(K)) \cap V(K)$ is dense in $(Y, f^{-1}M)(\mathbf{A}_K^T)$ for every place $v \in \Omega_K \setminus T$. □

Remark 2.2.15. The surjectivity condition in Proposition 2.2.14 is referred to as "arithmetic surjectivity" in the literature, see [LSS20].

2.2.3 Integral $\mathcal{M}$-approximation and the $\mathcal{M}$-Hilbert property

Now we will explore the relationship between integral $\mathcal{M}$-approximation, the $\mathcal{M}$-Hilbert property and Zariski density. This is an extension of the classical theory of the Hilbert property and weak approximation as presented in [Ser08, Chapter 3], and of the Campana version introduced in [NS24].

Definition 2.2.16. Let X be an integral variety over K and let $A \subset X(K)$. We say that A is of *type I* if $A \subset Z(K)$ for some proper closed subset Z of X.

We say that A is of *type II* if there is an integral variety Y with $\dim Y = \dim X$ and a generically finite morphism $f : Y \to X$ of degree ≥ 2 such that $A \subset f(Y(K))$.

We say that A is *thin* if it is a finite union of sets of type I and II.

Remark 2.2.17. We do not assume that the morphisms in Definition 2.2.16 are separable, unlike the definitions given in [BFP14; Lug22]. In particular, thin sets as considered in those articles are thin sets as defined here, but not vice versa. For example, if k is a perfect field of characteristic $p > 0$, then $k(t^p) \subset \mathbb{A}^1(k(t))$ is a thin set in the terminology of this thesis, but not according to the notion in [BFP14; Lug22].

Definition 2.2.18. Let $(\mathcal{X}, \mathcal{M})$ be an integral model over B of a pair (X, M). We say that $(\mathcal{X}, \mathcal{M})$ satisfies the *$\mathcal{M}$-Hilbert property over B* if $(\mathcal{X}, \mathcal{M})(B)$ is not thin as a subset of $X(K)$.

Note that every PF field K is Hilbertian (see for example [FJ05, Chapter 13]) and imperfect if it is of positive characteristic, so by [FJ05, Proposition 12.4.3], $\mathbb{A}^1(K)$ is an example of a set which is not thin.

In order to relate the $\mathcal{M}$-Hilbert property to integral $\mathcal{M}$-approximation over global function fields as we do in Theorem 1.1.1, we need the following lemma.

Lemma 2.2.19. *Let k be a field of positive characteristic and let $f\colon Y \to X$ be an inseparable generically finite morphism of integral varieties over $k(\!(t)\!)$, where X is smooth over $k(\!(t)\!)$. Then the image of the induced map $Y(k(\!(t)\!)) \to X(k(\!(t)\!))$ is a nowhere dense subset of $X(k(\!(t)\!))$, where the topologies are induced by the topology on $k(\!(t)\!)$.*

Proof. The proof is based on the observation that $k(\!(t)\!) \to k(\!(t)\!)$ given by $x \mapsto x^p$ has nowhere dense image in $k(\!(t)\!)$, where $p = \mathrm{char}(k)$. This is because the image is contained in the closed set $\bigcap_{i\in\mathbb{Z}, p\nmid i} C_i$, where C_i is the set of Laurent polynomials with vanishing i-th coefficient. Since every nonempty open set contains elements outside of this closed set, the image is nowhere dense in $k(\!(t)\!)$.

First we show it suffices to prove the statement for an open subvariety $U \subset X$. Since X is smooth, by [BLR90, §2.2, Proposition 11] it is covered by open subvarieties $U \subset X$ such that there exists étale morphism $U \to \mathbb{A}^d_{k(\!(t)\!)}$, where d is the dimension of X. By [Con12, Lemma 5.3] the induced map $U(k(\!(t)\!)) \to \mathbb{A}^d_{k(\!(t)\!)}(k(\!(t)\!))$ is a local homeomorphism. Since for a closed subvariety $V \subset \mathbb{A}^d_{k(\!(t)\!)}$, the subset $V(k(\!(t)\!)) \subset \mathbb{A}^d_{k(\!(t)\!)}(k(\!(t)\!))$ is nowhere dense, it follows that for any proper closed subvariety $V \subset X$, the subset $V(k(\!(t)\!)) \subset X(k(\!(t)\!))$ is nowhere dense. Therefore it suffices to prove the statement for some nonempty open subvariety $U \subset X$.

For a generically finite dominant morphism $f\colon Y \to X$ of integral varieties, we can factor the extension $K(Y)/K(X)$ as a separable extension $L/K(X)$ followed by a totally inseparable extension $K(Y)/L$, see [Stacks, Tag 030K]. This corresponds to factoring the morphism into rational maps $Y \dashrightarrow Z \dashrightarrow X$, where $Z \dashrightarrow X$ is separable and $Y \dashrightarrow Z$ is totally inseparable. We can assume that Z is not geometrically integral, as otherwise [Stacks, Tag 0CDW] implies that $Z(k(\!(t)\!))$ is not Zariski dense in Z. Since we can restrict X to a nonempty open U, without loss of generality, we can assume that the maps are morphisms and $Z \to X$ is étale. Therefore, by [Con12, Lemma 5.3] we can without loss of generality assume that $f\colon Y \to X$ is purely inseparable of degree equal to $p := \mathrm{char}(k)$.

Choose a separable generically finite rational map $X \dashrightarrow \mathbb{A}^d_{k(\!(t)\!)}$. Then the field extension $k(\!(t)\!)(x_1,\dots,x_d) \subset k(\!(t)\!)(Y)$ induced by this rational map has inseparable degree p. There are only finitely many fields in $k(\!(t)\!)(Y)$ containing $k(\!(t)\!)(x_1,\dots,x_d)$. Indeed, by Galois theory there are only finitely many separable extensions L of $k(\!(t)\!)(x_1,\dots,x_d)$ in $k(\!(t)\!)(Y)$. Furthermore, for any such a field extension L, there is at most a single nontrivial purely inseparable extension L'/L in $k(\!(t)\!)(Y)$, since given another such extension L'' the compositum $L''L'/L$ is purely inseparable and therefore has degree p, by multiplicativity of inseparable degrees [Stacks, Tag 09HK]. The primitive element theorem [Stacks, Tag 030N] implies that $k(\!(t)\!)(x_1,\dots,x_d) \subset k(\!(t)\!)(Y)$ is a simple extension. Therefore for some nonzero polynomial $g \in k(\!(t)\!)[y, x_1,\dots,x_d]$ which is separable in y,

$$k(\!(t)\!)(Y) = k(\!(t)\!)(x_1,\dots,x_d)[z]/(g(z^p, x_1,\dots,x_d)),$$

where the p-th power is present since the extension is not separable. The separable closure of $k(\!(t)\!)$ in this field is

$$k(\!(t)\!)(X) = k(\!(t)\!)(x_1, \dots, x_d)[y]/(g(y, x_1, \dots, x_d)).$$

In order to reduce the argument to the power map $x \mapsto x^p$, we construct separable dominant maps $Y \dashrightarrow \mathbb{A}^d_{k(\!(t)\!)}$ and $X \dashrightarrow \mathbb{A}^d_{k(\!(t)\!)}$. If $\frac{\mathrm{d}g}{\mathrm{d}x_d}$ is not the zero polynomial, then the Jacobi criterion implies that the projections to affine space corresponding to the inclusions

$$k(\!(t)\!)(y, x_1, \dots, x_{d-1}) \subset k(\!(t)\!)(X)$$

and

$$k(\!(t)\!)(z, x_1, \dots, x_{d-1}) \subset k(\!(t)\!)(Y)$$

are separable. If $\frac{\mathrm{d}g}{\mathrm{d}x_d}$ is the zero polynomial, we can reduce to the case that $\frac{\mathrm{d}g}{\mathrm{d}x_d}$ is not identically zero, by using the fact that g is separable in y and by using linear change of variables replacing y with $y + x_d$.

Therefore, we get a Cartesian diagram of dominant rational maps

$$\begin{tikzcd}
Y \arrow[r, dashed] \arrow[d, "f"] & \mathbb{A}^d_{k(\!(t)\!)} \arrow[d, "h"] \\
X \arrow[r, dashed] & \mathbb{A}^d_{k(\!(t)\!)},
\end{tikzcd}$$

where the horizontal maps are separable, and the map h is the p-th power map in the first coordinate and the identity in the other coordinates. Since we can without loss of generality restrict X to an open subset U, we can assume that the horizontal maps are étale morphisms. Since the image of $\mathbb{A}^d_{k(\!(t)\!)}(k(\!(t)\!)) \to \mathbb{A}^d_{k(\!(t)\!)}(k(\!(t)\!))$ is a nowhere dense subset, applying [Con12, Lemma 5.3] once more to the horizontal maps shows that the image of $Y(k(\!(t)\!))$ in $X(k(\!(t)\!))$ is nowhere dense. □

Now we will prove Theorem 1.1.1, which we will prove separately for global fields and for function fields over infinite fields.

Proof of Theorem 1.1.1 for global fields. First we prove the statements for global fields. This part of the proof is based on [NS24, Theorem 1.1, Remark 2.12], and generalizes it to global fields and varieties which are not normal or even integral.

We first assume that X is geometrically irreducible. Note that the statement of the theorem is equivalent to saying that if the $\mathcal{M}$-Hilbert property over B is not satisfied and $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$, then integral $\mathcal{M}$-approximation does not hold for any finite set of places T. We proceed by proving the following stronger claim: if $A \subset (\mathcal{X}, \mathcal{M})(B)$ is thin, then there is a finite set of places $T' \subset \Omega_K$ disjoint from T such that A is not dense in $\prod_{v \in T'} (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$. We recover the original statement by taking $A = (\mathcal{X}, \mathcal{M})(B)$. By the argument in [Ser08, Proof of Theorem 3.5.3] if the claim holds for thin sets $A_1, A_2 \in (\mathcal{X}, \mathcal{M})(B)$, then it holds for $A_1 \cup A_2$.

For a set A which is not Zariski dense (thin of type I), the result follows from the Lang–Weil bound [LW54] in the same manner as in the proof of [NS24, Theorem 1.1]. In particular, we conclude that $(\mathcal{X}, \mathcal{M})(B)$ is Zariski dense in X.

Now let $A \subset f(Y(K)) \cap (\mathcal{X}, \mathcal{M})(B)$ for Y an integral variety over K and $f \colon Y \to X$ a dominant generically finite morphism of degree ≥ 2. Without loss of generality, we can assume that f is finite, since the image of a Zariski closed subset in Y is a type I thin set. There are two cases to consider: either f is separable or it is inseparable. If f is separable, then the result follows from [Ser08, Theorem 3.6.2] (which extends to separable morphisms over global fields) and the proof of [NS24, Theorem 1.1], see also [NS24, Remark 2.12]. This finishes the claim for separable morphisms. For global function fields we need to consider inseparable morphisms.

Thus we now assume $f: Y \to X$ is an inseparable morphism. For every place $v \in \Omega_K$, the Cohen structure theorem [Stacks, Tag 0C0S] implies $\mathcal{O}_v \cong k_v[\![t]\!]$. Therefore Lemma 2.2.19 implies that for every place $v \in \Omega_K$ the subset $f(Y(K_v)) \cap U(K_v) \subset U(K_v)$ is nowhere dense, where U is the smooth locus of X. Since $(\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v)$ is nonempty and open in $X(K_v)$ by Proposition 2.1.2, $A \cap U(K_v) \cap (\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v) \subset (\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v)$ is nowhere dense. Since $X(K_v) \setminus U(K_v)$ is a thin set of type I, its restriction to $(\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v)$ is not dense in $(\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v)$. Thus $A \cap (\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v) \subset (\mathcal{X}, \mathcal{M}_{\mathrm{fin}})(\mathcal{O}_v)$ is not dense so $A \subset (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ is not dense. This proves the claim.

Now assume that X is irreducible, but not necessarily geometrically irreducible. We will prove that $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ contains a smooth point in $X(K_v)$ for some place $v \in \Omega_K$, and then conclude that therefore $X(K)$ contains a smooth point. As X is irreducible, $U = X \setminus \bigcup_{\alpha \in \mathcal{A}} D_\alpha$ is irreducible as well. Write $\mathcal{U} = \mathcal{X} \setminus \bigcup_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$. If X is not geometrically irreducible, choose a finite separable extension K' of K such that $X_{K'}$ splits into geometrically irreducible components X_i. Write B' for the regular scheme associated to the integral closure of $\mathcal{O}_B$ in K'. The splitting also induces a splitting of the integral model $\mathcal{X} \times_B B'$ into components $\mathcal{X}_i$ which are integral models of the X_i. Write $\mathcal{U}_i = \mathcal{X}_i \cap (\mathcal{U} \times_B B')$. Write T' for the places in $\Omega_{K'}$ above T. As in the previous part, the Lang–Weil bound [LW54] implies that, for all but finitely many places $v' \in \Omega_{K'}$, $\mathcal{U}_i(k_{v'})$ contains a smooth point. In particular, if we choose a finite place $v \in B \setminus T$ which splits completely in K' and such that for a place $v' \in B'$ above v, $\mathcal{U}_i(k_{v'})$ contains a smooth point, then $\mathcal{U}(k_v) = \mathcal{U}(k_{v'})$ contains a smooth point as it contains $\mathcal{U}_i(k_{v'})$. Thus by Hensel's lemma [Poo17, Theorem 3.5.63] $\mathcal{U}(\mathcal{O}_v)$ contains a smooth point. Since $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$ and $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T, we see that $X(K)$ contains a smooth point, and thus by [Stacks, Tag 0CDW], we see that X is geometrically integral.

If X is not irreducible, then there are disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$, where $\mathcal{U}$ is as above. By the same reasoning as when X was irreducible, the Lang–Weil bound implies that we can find distinct places $v_1, v_2 \in \Omega_K \setminus T$ such that $\mathcal{U}_1(\mathcal{O}_{v_1})$ and $\mathcal{U}_2(\mathcal{O}_{v_2})$ are nonempty. But $P \in (\mathcal{X}, \mathcal{M})(B)$ cannot simultaneously lie in the open sets $\mathcal{U}_1(\mathcal{O}_{v_1})$ and $\mathcal{U}_2(\mathcal{O}_{v_2})$ as $\mathcal{U}_1$ is disjoint from $\mathcal{U}_2$. This is a contradiction, so we find that X has to be irreducible. □

Proof of Theorem 1.1.1 for function fields over infinite fields. Now assume that $K = k(C)$ for an infinite field k and a regular curve C over k. First we assume that X is irreducible. Let $\mathcal{U} = \mathcal{X} \setminus \bigcup_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$ be the closure of $U = X \setminus \bigcup_{\alpha \in \mathcal{A}} D_\alpha$ in $\mathcal{X}$. Since X is geometrically reduced, there exists a nonempty open subset $\mathcal{Y} \subset \mathcal{U}$ such that the restriction of the structure morphism $\mathcal{X} \to B$ to $\mathcal{Y}$, written $f: \mathcal{Y} \to B$, is smooth. Note that B is a Jacobson scheme [Stacks, Tag 01P2], f is dominant and of finite presentation, and by [Mor20, Théorème 4.2.3.(1)] B is a "schéma de Poonen". Therefore [Mor20, Théorème 3.2.(3)] implies that there exists a point $y \in \mathcal{Y}$ such that $f(y)$ is a closed point and the residue field satisfies $k(y) = k(f(y))$. If we write $v = f(y)$, then this implies that $y \in \mathcal{Y}(k_v) \subset \mathcal{U}(k_v)$ is a rational k_v-point, and thus by Hensel's Lemma [Poo17, Theorem 3.5.63], $\mathcal{U}(\mathcal{O}_v)$ contains a smooth point P. In particular, [Stacks, Tag 0CDW] implies that X is geometrically integral.

Since P is smooth, there exists an open $V \subset X$ which is smooth over K_v. Thus Proposition 2.2.6 implies $V \cap \mathcal{U}(\mathcal{O}_v)$ is Zariski dense as it contains P, so $\mathcal{U}(\mathcal{O}_v)$ is Zariski dense. Since $(\mathcal{X}, \mathcal{M})(B)$ is dense in $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ in the analytic topology, it is Zariski dense in X.

Now if we assume that X is not irreducible, we find a contradiction in the exact same manner as in the global function field case. So X has to be geometrically integral. □

The next example shows that the set of rational points on a variety that satisfies weak approximation can fail to be Zariski dense if the variety is not geometrically reduced.

Example 2.2.20. Let $k = \mathbb{F}_p(a, b)$ and $K = k(t)$ for algebraically independent variables a, b, t. Define X to be the closed subvariety of $\mathbb{P}^3_k(x, y, z, w)$ given by

$$x^p - z^p a = y^p - z^p b = 0.$$

For every place of v of K, K_v is a simple extension of $k(\!(t)\!)$. However, $k(\!(t)\!)(a^{1/p}, b^{1/p})$ is a degree p^2 extension of $k(\!(t)\!)$ which is not simple. Indeed if it were simple, then there would be a primitive element α such that $k(\!(t)\!)(a^{1/p}, b^{1/p}) = k(\!(t)\!)(\alpha)$, but $\alpha^p \in k(\!(t)\!)$ would imply that the extension has degree p rather than p^2. Thus we see that K_v cannot contain $k(\!(t)\!)(a^{1/p}, b^{1/p})$. Consequently, for any K_v-point on X we must have $z = 0$, and therefore

$$X(K_v) = X(K) = X(k) = \{(0:0:0:1)\}.$$

This implies that X satisfies weak approximation while $X(K)$ is not Zariski dense.

As a consequence of Theorem 1.1.1, we give a new proof of Minchev's theorem.

Corollary 2.2.21 ([Min89, Theorem 1])**.** *Let U be a normal variety over a number field K, which satisfies strong approximation off a finite set of places $T \subset \Omega_K$. Then $U_{\overline{K}}$ is algebraically simply connected, i.e. $\pi_1(U_{\overline{K}}) = \{1\}$.*

Proof. Let X be a normal compactification of U and let $\mathcal{X}$ be a normal integral model of X over $\mathcal{O}_S$, where $T \subset S$ is some finite set of places. Let $\mathcal{D}$ be the Zariski closure of $X \setminus U$ in $\mathcal{X}$ and let $\mathcal{U} = \mathcal{X} \setminus \mathcal{D}$ be its complement. By Proposition 2.2.12, $\mathcal{U}$ satisfies integral strong approximation off S. Therefore, Theorem 1.1.1 implies that X is geometrically integral and $\mathcal{U}(\mathcal{O}_S) \subset X(K)$ is not thin. Assume that $U_{\overline{K}}$ is not simply connected. Then [Poo17, Lemma 3.5.57] implies that there exists a nontrivial finite étale morphism $f\colon Y \to U$ where Y is a geometrically integral variety over K. By spreading out [Poo17, Theorem 3.2.1(ii)], there exists a finite set of places S' containing S and a finite étale morphism $\mathcal{Y} \to \mathcal{U} \times_{\mathcal{O}_S} \operatorname{Spec} \mathcal{O}_{S'}$ extending f, where $\mathcal{Y}$ is a scheme with $\mathcal{Y}_K \cong Y$. This contradicts [Lug22, Theorem 1.8], so $U_{\overline{K}}$ has to be simply connected. □

2.3 $(M, \mathcal{M}')$-approximation

We conclude Chapter 2 by generalizing the notions of adelic M-points and integral adelic $\mathcal{M}$-points introduced in Definition 2.2.1, in order to relate these notions to the adelic points considered in [MNS24]. The notions and results in this section will not be used in later chapters.

Definition 2.3.1. Let (K, C) be PF field, $T \subset \Omega_K$ be a finite sets of places and let $B \subset C$ be an open subscheme. Let $(X, M) \subset (X, M')$ be an inclusion of pairs with integral models $(\mathcal{X}, \mathcal{M}) \subset (\mathcal{X}, \mathcal{M}')$. We define the space of *adelic* $(\mathcal{M}, \mathcal{M}')$*-points over* B *prime to* T to be the restricted product

$$(\mathcal{X}, \mathcal{M}, \mathcal{M}')(\mathbf{A}_B^T) = \prod_{v \in \Omega_K \setminus T} ((\mathcal{X}, \mathcal{M}')(\mathcal{O}_v) \cap (X, M')(K_v), (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)).$$

If $(\mathcal{X}, \mathcal{M}) = (\mathcal{X}, \mathcal{M}')$, this recovers the notion of integral adelic $\mathcal{M}$-points, while if $(\mathcal{X}, \mathcal{M}')(\mathcal{O}_v) = (X, M)(K_v)$ for all $v \in \Omega_K$ then this recovers the notion of adelic M-points.

Remark 2.3.2. The definition of $(\mathcal{M}, \mathcal{M}')$-points generalizes the notion of adelic semi-integral points considered in [MNS24]: if $\mathcal{M}'$ encodes the Campana (respectively Darmon) condition for a divisor $\mathcal{D}_\mathbf{m}$ as in Definition 2.1.19 and $\mathcal{M}$ is the integrality condition for the support of $\mathcal{D}_\mathbf{m}$, then there is an equality of topological spaces

$$(\mathcal{X}, \mathcal{M}, \mathcal{M}'_{\mathrm{fin}})(\mathbf{A}_B^T) = (\mathcal{X}, \mathcal{D}_\mathbf{m})^*_{\mathrm{st}}(\mathbf{A}_B^T),$$

where the right hand side is the set of strict T-adelic semi-integral points as in [MNS24, Definition 2.15]. However, the full set of T-adelic semi-integral points $(\mathcal{X}, \mathcal{D}_\mathbf{m})^*(\mathbf{A}_B^T)$ differs as a set from $(\mathcal{X}, \mathcal{M}, \mathcal{M}')(\mathbf{A}_B^T)$, since a point $(P_v)_v \in (\mathcal{X}, \mathcal{M}, \mathcal{M}')(\mathbf{A}_B^T)$ is integral with respect to the support of $\mathcal{D}_\mathbf{m}$ at all but finitely many places v, while a non-strict point in $(\mathcal{X}, \mathcal{D}_\mathbf{m})^*(\mathbf{A}_B^T)$ lies in the boundary for all places.

In the remainder of the section we will generalize some results of Section 2.2.

The following proposition is an analogue of Proposition 2.2.2, and shows that the space of adelic $(\mathcal{M}, \mathcal{M}')$-points does not depend on the choice of an integral model for (X, M).

Proposition 2.3.3. *If* (X, M) *is a pair with integral models* $(\mathcal{X}, \mathcal{M})$ *and* $(\mathcal{X}, \mathcal{M}')$ *over* B*, and* $(\mathcal{X}, \mathcal{M}'')$ *is an integral model over* B *of the pair* (X, M) *such that* $(\mathcal{X}, \mathcal{M}), (\mathcal{X}, \mathcal{M}') \subset (\mathcal{X}, \mathcal{M}'')$ *are open, then there is a canonical homeomorphism*

$$(\mathcal{X}, \mathcal{M}, \mathcal{M}'')(\mathbf{A}_B^T) \cong (\mathcal{X}, \mathcal{M}', \mathcal{M}'')(\mathbf{A}_B^T).$$

Proof. This follows from the proof of Proposition 2.2.2 by replacing the term $(X, M)(K_v)$ in the proof with $(\mathcal{X}, \mathcal{M}')(\mathcal{O}_v)$. □

In light of the previous proposition, and the fact that every pair over K has an integral model over C, we will not specify the integral model $(\mathcal{X}, \mathcal{M})$ of (X, M) and write $(M, \mathcal{M}')$ instead of $(\mathcal{M}, \mathcal{M}')$.

The next proposition is an analogue of Proposition 2.2.12 and gives conditions for when the natural inclusion maps are embeddings or have dense image.

Proposition 2.3.4. *Let $(X, M) \subset (X, M') \subset (X, M'')$ be pairs and let $(\mathcal{X}, \mathcal{M}) \subset (\mathcal{X}, \mathcal{M}') \subset (\mathcal{X}, \mathcal{M}'')$ be inclusions of integral models over B of the respective pairs, and let $T \subset \Omega_K$ be a finite set of places. Then:*

1. *The natural inclusion $(\mathcal{X}, M, \mathcal{M}'')(\mathbf{A}_B^T) \hookrightarrow (\mathcal{X}, M', \mathcal{M}'')(\mathbf{A}_B^T)$ has dense image if $(\mathcal{X}, M, \mathcal{M}'')(\mathbf{A}_B^T) \neq \emptyset$.*
2. *The natural inclusion $(\mathcal{X}, M, \mathcal{M}')(\mathbf{A}_B^T) \hookrightarrow (X, M, \mathcal{M}'')(\mathbf{A}_B^T)$ is a topological embedding, and it is open if $(\mathcal{X}, \mathcal{M}')(\mathcal{O}_v) \subset (\mathcal{X}, \mathcal{M}'')(\mathcal{O}_v)$ is open for all $v \in \Omega_K \setminus T$.*

Proof. This is a direct consequence of Proposition 1.3.6. $\square$

We now generalize the notion M-approximation to $(M, \mathcal{M}')$-approximation.

Definition 2.3.5. Let $T \subset \Omega_K$ be a finite set of places, let $(X, M), (X, M')$ be pairs over (K, C) such that $M \subset M'$ and let $(\mathcal{X}, \mathcal{M}')$ be an integral model of (X, M') over $B \subset C$. Then we say that $\mathcal{X}$ satisfies $(M, \mathcal{M}')$-*approximation off* T if the image of the natural inclusion

$$(\mathcal{X}, \mathcal{M}')(B) \cap (X, M)(K) \hookrightarrow (\mathcal{X}, M, \mathcal{M}')(\mathbf{A}_B^T)$$

is dense.

The restriction to $(X, M)(K)$ ensures that the inclusion map is well-defined.

Example 2.3.6. If $(\mathcal{X}, \mathcal{M}')$ is the pair corresponding to the Campana (or Darmon) condition for the Campana pair $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ and $M \subset M'$ is the integral condition with respect to the support of $\mathcal{D}_\mathbf{m}$, then $(M, \mathcal{M}')$-approximation on X coincides with strong Campana (or Darmon) approximation on $(\mathcal{X}, \mathcal{D}_\mathbf{m})$ as studied in [MNS24], up to the slightly differing adelic space as discussed in Remark 2.3.2.

3. Split toric varieties and M-approximation

In this chapter we study M-approximation and the $\mathcal{M}$-Hilbert property for toric pairs.

3.1 Cox coordinates on toric varieties

In this section we introduce toric varieties and their Cox coordinates, following and generalizing [CLS11, Chapter 5.1] and [Sal98, Chapter 8]. Given any fan Σ as in [Sal98, Definition 8.1.1] (not necessarily complete or regular), we define the toric scheme associated to Σ over $\mathbb{Z}$ to be $\mathcal{X}_{\Sigma,\mathbb{Z}}$ as in [Sal98, Remark 8.6]. This is a normal, separated scheme over $\mathbb{Z}$ and it is proper (resp. smooth) over $\mathbb{Z}$ if and only if Σ is complete (resp. regular). For any scheme S, we define the toric scheme associated to Σ over S to be $\mathcal{X}_{\Sigma,S} := \mathcal{X}_\Sigma \times_{\mathbb{Z}} S$. For the remainder of the section, we let K be a field and write $X_\Sigma := \mathcal{X}_{\Sigma,K}$ for the normal split toric variety associated to the fan Σ.

For a fan Σ, we write $\Sigma(1)$ for the collection of rays in Σ, $\Sigma_{\max}$ for the collection of maximal cones in Σ, and $\{D_1,\dots,D_n\}$ for the set of torus-invariant prime divisors on $X := X_\Sigma$. We denote the lattice of cocharacters of $\mathcal{X}$ by N and its dual by $N^\vee$. For a torus-invariant divisor D_i, we write $\rho_i \subset N_\mathbb{R}$ for the associated ray and $n_{\rho_i} \in N$ for its ray generator.

Definition 3.1.1. If $X = X_\Sigma$ is a normal split toric variety over a PF field (K, C), then its *toric integral model* over $B \subset C$ is $\mathcal{X} = \mathcal{X}_{\Sigma,B}$. If (X, M) is a pair with $\mathcal{A} = \{1,\dots,n\}$ and $D_1,\dots,D_n$ the torus-invariant prime divisors on X, then we call (X, M) a *toric pair* and we say that its *toric integral model* over B is $(\mathcal{X}_{\Sigma,B}, \mathcal{M}^c)$. We denote the open torus in X by U.

In the remainder of this chapter, we assume that the ray generators n_{ρ_i} span $N_\mathbb{R}$. This is equivalent to the split toric variety $X = X_{\Sigma,K}$ not having torus factors, or equivalently $\mathcal{O}_X(X_{\overline{K}})^\times = \overline{K}^\times$.

Now we introduce Cox coordinates on the integral points on the toric variety: we write the toric scheme $\mathcal{X} = \mathcal{X}_{\Sigma,\mathbb{Z}}$ as a quotient $\mathcal{X} = \mathcal{Y}/\mathcal{G}$ for some open subscheme $\mathcal{Y} \subset \mathbb{A}^n_\mathbb{Z}$ and a group scheme $\mathcal{G} \subset \mathbb{G}^n_{m,\mathbb{Z}}$. We have an exact sequence

$$0 \to N^\vee \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(\Sigma) \to 0, \tag{3.1.1}$$

as in [CLS11, Theorem 4.1.3], where $\mathrm{Cl}(\Sigma)$ is the class group of any toric variety over a field with the fan Σ. We define the $\mathbb{Z}$-group scheme

$$\mathcal{G} = \mathcal{H}om(\mathrm{Cl}(\Sigma), \mathbb{G}_{m,\mathbb{Z}}),$$

which by the above exact sequence is the kernel of the homomorphism

$$\mathbb{G}_{m,\mathbb{Z}}^n \cong \mathcal{H}om(\mathbb{Z}^{\Sigma(1)}, \mathbb{G}_{m,\mathbb{Z}}) \to \mathbb{G}_{m,\mathbb{Z}}^d \cong \mathcal{H}om(N^\vee, \mathbb{G}_{m,\mathbb{Z}})$$

induced by the inclusion $N^\vee \hookrightarrow \mathbb{Z}^{\Sigma(1)}$. Here d is the rank of the lattice N. In particular, for every ring R, we have the description

$$\mathcal{G}(R) = \left\{ \mathbf{t} \in (R^\times)^n \,\middle|\, \prod_{i=1}^n t_i^{\langle m, n_{\rho_i}\rangle} = 1 \text{ for all } m \in N^\vee \right\},$$

where $\langle \cdot, \cdot \rangle \colon N^\vee \times N \to \mathbb{Z}$ is the natural pairing. Now for each cone $\sigma \in \Sigma$, let

$$x^{\hat{\sigma}} = \prod_{\substack{i=1 \\ \rho_i \not\subset \sigma}}^n x_i,$$

and let

$$\mathcal{Z} = \{x^{\hat{\sigma}} = 0 \mid \text{for all } \sigma \in \Sigma\} \subset \mathbb{A}_{\mathbb{Z}}^n.$$

Then $\mathcal{Y} = \mathbb{A}_{\mathbb{Z}}^n \setminus \mathcal{Z}$ carries the natural structure of a toric scheme, and the subscheme $\mathcal{G}$ acts on it by coordinate-wise multiplication. Similarly to [CLS11, Proposition 5.1.9], we find the *Cox morphism* $\pi \colon \mathbb{A}_{\mathbb{Z}}^n \setminus \mathcal{Z} \to \mathcal{X}$ of toric schemes, which is constant on $\mathcal{G}$-orbits and gives a bijection between the closed toric subschemes of $\mathcal{Y}$ and those of $\mathcal{X}$.

Lemma 3.1.2. *Let $\mathcal{X}$, $\mathcal{Y}$ and π be as in the discussion above. The morphism $\pi \colon \mathcal{Y} \to \mathcal{X}$ is an universal categorical quotient for the action of $\mathcal{G}$ on $\mathbb{A}_{\mathbb{Z}}^n \setminus \mathcal{Z}$. If furthermore Σ is regular and* $\mathrm{Cl}(\Sigma)$ *is torsion-free, then π is a $\mathcal{G}$-torsor.*

If Σ is regular, the Cox morphism is referred to as the universal torsor of $\mathcal{X}$, such as in [Sal98; FP16]. To prove the lemma we first need the following generalization of [CLS11, Proposition 5.0.9] to general rings.

Proposition 3.1.3. *Let $\mathcal{G}$ be a linearly reductive group scheme over a ring R (as in [Alp13, Section 12]) acting on an affine R-scheme* $\operatorname{Spec} A$*. Then the induced morphism* $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathcal{G}}$ *is a universal categorical quotient for this action, where $A^{\mathcal{G}}$ is the subalgebra containing the elements in A invariant under the action of $\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is linearly reductive, by [Alp13, Remark 4.8] (see also [Alp13, Theorem 13.2]) the morphism of R-stacks $[\operatorname{Spec} A/\mathcal{G}] \to \operatorname{Spec} A^{\mathcal{G}}$ is a good moduli space [Alp13, Definition 4.1]. In particular, any morphism $\operatorname{Spec} A \to S$ to some R-scheme S which is constant on $\mathcal{G}$-orbits factors uniquely as $\operatorname{Spec} A \to [\operatorname{Spec} A/\mathcal{G}] \to \operatorname{Spec} A^{\mathcal{G}} \to S$, since good moduli spaces are universal for maps to schemes [Alp13, Theorem 4.16(vi)]. Thus we see that $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathcal{G}}$ is a categorical quotient. Since good moduli spaces are preserved under base change [Alp13, Proposition 4.7], it is furthermore a universal categorical quotient. □

Proof of Lemma 3.1.2. The proof of the first part is exactly as in [CLS11, Theorem 5.1.11], where we replace the use of [CLS11, Proposition 5.0.9] by Proposition 3.1.3, where we use that $\mathcal{G}$ is linearly reductive by [Alp13, Example 12.4.(2)]. The second part follows from [CLS11, Theorem 3.3.19], where regularity is used to ensure that the bijective morphisms of cones $\hat{\sigma} \to \sigma$ induced by π restrict to bijective maps $\hat{\sigma} \cap \left(\mathbb{Z}^{\Sigma(1)}\right)^\vee \to \sigma \cap N$. □

Using the construction just given, we can define Cox coordinates on a split toric variety.

Definition 3.1.4. Let $\mathcal{X}$ and $\pi\colon \mathcal{Y} \to \mathcal{X}$ be as above, and let B be a scheme. For any $P \in \mathcal{Y}(B)$, we say that $P = (P_1, \dots, P_n)$ are *Cox coordinates* for the point $\pi(P) \in \mathcal{X}(B)$, where $P_i \in \mathcal{O}(B)$. We will write $\pi(P) = (P_1 : \cdots : P_n)$, in analogy with homogeneous coordinates.

When can all points in $\mathcal{X}(B)$ be represented by Cox coordinates for a scheme B? If Σ is regular and complete, then $\mathrm{Cl}(\Sigma)$ is torsion-free. Therefore Lemma 3.1.2 shows that the Cox morphism is a $\mathcal{G}$-torsor. Therefore [FP16, Proposition 2.1] gives a decomposition

$$\mathcal{X}(B) = \bigsqcup_{[W] \in H^1_{fppf}(B, \mathcal{G})} {}_W\pi({}_W\mathcal{Y}(B)), \tag{3.1.2}$$

for every scheme B. Here ${}_W\pi\colon {}_W\mathcal{Y} \to \mathcal{X}$ is the twist of π by $-[W] \in H^1_{fppf}(B, \mathcal{G})$ as defined in [Sko01, p.22].

Note that in this case $\mathcal{G} \cong \mathbb{G}_{m,B}^{n-d}$. Thus $H^1_{fppf}(B, \mathcal{G}) \cong \mathrm{Pic}(B)^{n-d}$ if B is regular. If $B = \operatorname{Spec} R$ for a unique factorisation domain then this implies that every R-point is represented by Cox coordinates.

Proposition 3.1.5. *Let Σ be a regular and complete fan, let $\mathcal{X} = \mathcal{X}_\Sigma$, and let R be a unique factorisation domain. Then every $P \in \mathcal{X}(R)$ is represented by Cox coordinates:*

$$\mathcal{X}(R) = \pi(\mathcal{Y}(R)), \tag{3.1.3}$$

where π is the Cox morphism.

If we instead consider singular toric varieties, then we can still show that rational points are represented by Cox coordinates, as long as $\mathrm{Cl}(\Sigma)$ is torsion-free.

Proposition 3.1.6. *Let Σ be a fan such that $\mathrm{Cl}(\Sigma)$ is torsion-free and let k be a field. Then every point $P \in \mathcal{X}(k) = \mathcal{X}_\Sigma(k)$ is represented by Cox coordinates:*

$$\mathcal{X}(k) = \pi(\mathcal{Y}(k)), \tag{3.1.4}$$

where π is the Cox morphism.

Proof. By the Orbit-Cone Correspondence [CLS11, Theorem 3.2.6, Proposition 3.2.7], the torus-invariant orbits $V \cong \mathbb{G}_m^s(k)$ on a split toric scheme $\mathcal{X}$ correspond to the closed toric subschemes $\mathcal{Z}$ of $\mathcal{Y}$ of dimension s by restricting to the dense torus in $\mathcal{Z}$ and by taking k-points. Since π induces a bijection between the closed toric subschemes of $\mathcal{Y}$ and those of $\mathcal{X}$, it also induces a bijection between the torus orbits in $\mathcal{Y}(k)$ and in $\mathcal{X}(k)$. Since every torus orbit contains a k-point, every torus orbit $V \subset \mathcal{X}(k)$ contains the image of a point in $\mathcal{Y}(k)$. Since $\mathrm{Cl}(\Sigma)$ is torsion-free, the short exact sequence $0 \to N^\vee \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(\Sigma) \to 0$ splits, which gives a splitting of the associated exact sequence

$$0 \to \mathcal{G} \to \mathbb{G}_{m,\mathbb{Z}}^n \to \mathbb{G}_{m,\mathbb{Z}}^d \to 0,$$

and thus the map $\mathbb{G}_{m,\mathbb{Z}}^n(k) \to \mathbb{G}_{m,\mathbb{Z}}^d(k)$ is surjective. Let $V \subset \mathcal{X}(k)$ be a torus orbit and consider the map $\phi\colon \mathbb{G}_{m,\mathbb{Z}}^d(k) \to V$ from the k-points of the dense torus in $\mathcal{X}$

to V, induced by the map on tori. By combining [CLS11, Lemma 3.2.5] and the same splitting argument as above to the exact sequence (3.2.6) in [CLS11], we see that ϕ is surjective. Combining the surjectivity of these maps, the composite map $\mathbb{G}_{m,\mathbb{Z}}^n(k) \to V$ is surjective. By the commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{Z}}^n(k) & \longrightarrow & V' \\ \downarrow & & \downarrow \\ \mathbb{G}_{m,\mathbb{Z}}^d(k) & \xrightarrow{\phi} & V, \end{array}$$

where V' is the torus orbit in $\mathcal{Y}(k)$ above V, the map $V' \to V$ is surjective. Since every rational point lies in a torus orbit, the map $\mathcal{Y}(k) \to \mathcal{X}(k)$ is surjective. $\square$

By applying Proposition 3.1.6, we will show that any toric resolution of singularities of normal split toric varieties is surjective on rational points. For this we first need the following proposition, which characterizes when the class group of a split toric variety is torsion-free.

Proposition 3.1.7. *Let X be a normal split toric variety without torus factors. Then* $\mathrm{Cl}(X)$ *is torsion-free if and only if the ray generators $\{n_\rho \mid \rho \in \Sigma(1)\}$, span N as a lattice. Similarly, for a prime number p,* $\mathrm{Cl}(X)$ *does not have p-torsion if and only if the ray generators $\{n_\rho \mid \rho \in \Sigma(1)\}$ span N/pN as a lattice.*

Proof. The class group being torsion-free is equivalent to $\mathrm{Cl}(X)$ being a projective $\mathbb{Z}$-module, which by the splitting lemma is equivalent to the existence of a retraction $\mathbb{Z}^{\Sigma(1)} \to N^\vee$ of the exact sequence (3.1.1). Taking $\mathbb{Z}$-duals, we see this is equivalent to the projection $\mathbb{Z}^{\Sigma(1)} \to N$ having a section, and therefore to it being surjective. The second statement follows from tensoring with $\mathbb{Z}/p\mathbb{Z}$ and using the same argument for $\mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{Z}$. $\square$

Corollary 3.1.8. *Let k be a field and let X be a normal split toric variety over k. Then any proper birational toric morphism $f\colon \widetilde{X} \to X$ induces a surjection $\widetilde{X}(k) \to X(k)$. In particular, this holds whenever f is a toric resolution of singularities.*

Proof. Let Σ and $\widetilde{\Sigma}$ be the fans of X and $\widetilde{X}$ in a common co-character lattice N and let $P \in X(k)$ be a point, not contained in the open torus. We first show that we can assume that $\mathrm{Cl}(X)$ is torsion free. We compactify X to obtain a complete toric variety $X \subset X'$. Since the intersection of all toric affine opens $U_\sigma \subset X'$ corresponding to maximal cones $\sigma \in \Sigma$ is just the torus, there exists such an open U_σ not containing P. We subdivide σ into a collection of smooth maximal cones including σ'' and let X'' be the resulting complete toric variety. This yields a birational morphism $X'' \to X'$ of complete toric varieties, such that X'' contains an affine open $U_{\sigma''}$ corresponding to a smooth maximal cone. Therefore Proposition 3.1.7 implies that the class group of X'' is torsion-free. Since the statement to be proved is Zariski local, and there exists an open subset $P \in U \subset X$ such that the restriction $\tilde{X} \times_{X'} U \to U$ is an isomorphism, we can assume without loss of generality that $\mathrm{Cl}(X)$ is torsion free.

The toric birational morphism $\tilde{X} \to X$ is induced by subdividing the cones in Σ into smaller cones, which gives an inclusion $\Sigma(1) \subset \widetilde{\Sigma}(1)$ of sets of rays. Let $\widetilde{\Sigma}(1) \setminus \Sigma(1) = \{\rho_{n+1}, \ldots, \rho_{\tilde{n}}\}$ be the rays in $\widetilde{\Sigma}$ which do not lie in Σ. By Proposition

3.1.6 there exists Cox coordinates for P: $P = (a_1 : \cdots : a_n) \in X(K)$. Then consider the point $\tilde{P} = (a_1 : \cdots : a_n : 1 \cdots : 1) \in \widetilde{X}(k)$. Then it follows from the construction of Cox coordinates that $f(\widetilde{P}) = P$. □

3.2 M-approximation for split toric varieties

Now we restrict ourselves to the case where X is a complete normal split toric variety of dimension d over a PF field K. Consider a toric pair (X, M), where $M = ((D_i)_{i=1}^n, \mathfrak{M})$, and let $(\mathcal{X}, \mathcal{M})$ be its toric integral model. These assumptions are fixed for the rest of the paper, unless specified otherwise.

To understand whether a toric pair (X, M) satisfies M-approximation, we can use Proposition 2.2.14 to reduce to the case of a smooth toric variety:

Corollary 3.2.1. *Let (K, C) be a PF field, let $T \subset \Omega_K$ be a finite set of places, let (X, M) be a toric pair and let $f\colon \widetilde{X} \to X$ be a birational toric morphism of complete normal split toric varieties over K. Then (X, M) satisfies M-approximation off T if and only if $(\widetilde{X}, f^{-1}M)$ satisfies $f^{-1}M$-approximation off T.*

Proof. This follows directly from combining Proposition 2.2.14 with Corollary 3.1.8. □

3.2.1 Monoids in Theorem 1.1.3

In this section we introduce the monoids N_M, N_M^+ and $\rho(K, C)$, which are used in Theorem 1.1.3 and Theorem 3.3.5. These monoids indicate how $(\mathcal{X}, \mathcal{M})(B)$ is distributed in $X(K)$.

Definition 3.2.2. Let (X, M) be a toric pair where X is a complete smooth split toric variety. Define the homomorphism of monoids

$$\phi\colon \mathbb{N}^n \to N$$
$$(m_1, \ldots, m_n) \mapsto \sum_{i=1}^n m_i n_{\rho_i}.$$

Define the sublattice

$$N_M = \langle \phi(\mathbf{m}) \mid \mathbf{m} \in \mathfrak{M}_{\mathrm{fin,red}} \rangle \subset N,$$

and the submonoid N_M^+ generated by nonnegative linear combinations of the same elements.

The restriction to the reduced part reflects the fact that for every finite place $v \in \Omega_K$ and a point $P \in X(K_v)$, $\mathrm{mult}_v(P)$ lies in $\mathfrak{M}_{\mathrm{red}}$.

Remark 3.2.3. The monoid N_M^+ is equal to N_M if and only if the cone $N_{M,\mathbb{R}}^+$ generated by N_M^+ is equal to $\mathbb{R}^d$. This will be used in the proof of Theorem 1.1.3.

For each finite place v, we write

$$\phi_v := \phi \circ \mathrm{mult}_v \colon U(K_v) \to N. \tag{3.2.1}$$

The map $\phi_v \colon U(K_v) \to N$ is a group homomorphism, as the next proposition shows.

Proposition 3.2.4. *Let X be a smooth, complete split toric variety. For each finite place v, we write*

$$\phi_v := \phi \circ \mathrm{mult}_v \colon U(K_v) \to N. \tag{3.2.2}$$

The map $\phi_v \colon U(K_v) \to N$ defined above is a surjective group homomorphism with kernel $\mathcal{U}(\mathcal{O}_v)$, where $\mathcal{U} \cong \mathbb{G}_m^d$ is the open torus in $\mathcal{X}$. The homomorphism can be given in Cox coordinates as

$$\phi_v(u_1\pi^{w_1} : \cdots : u_n\pi^{w_n}) = \sum_{i=1}^{n} w_i n_{\rho_i},$$

where $w_i \in \mathbb{Z}$, $u_i \in \mathcal{O}_v^\times$ and $\pi \in \mathcal{O}_v$ is a uniformizer. This homomorphism gives a splitting

$$U(K_v) \cong \mathcal{U}(\mathcal{O}_v) \oplus N.$$

Furthermore, if $\widetilde{X}$ is a smooth complete split toric variety and $f \colon \widetilde{X} \to X$ is a toric morphism corresponding to the morphism of lattices $\overline{f} \colon \widetilde{N} \to N$, then we have $\phi_v \circ f = \overline{f} \circ \phi_v$. The map ϕ_v on the left corresponds to the map on the points in $X(K_v)$ and the map ϕ_v on the right corresponds to the map on the points in $\widetilde{X}(K_v)$.

As a consequence if f is birational, then for any toric pair (X, M) there are equalities of monoids $N_M^+ = N_{f^*M}^+$ and $N_M = N_{f^*M}$.

Proof. Let $P \in X(K_v)$ be a point. Since $\mathcal{X}_\Sigma$ has an open cover of affine toric schemes $\mathcal{V}_\sigma = \mathbb{A}^d$ corresponding to maximal cones $\sigma \in \Sigma$, $P \in \mathcal{V}_\sigma(\mathcal{O}_v)$ is satisfied for some maximal cone $\sigma \in \Sigma$. Thus we can represent the the point with Cox coordinates $P = (p_1 : \cdots : p_n)$ such that $p_i = 1$ if ρ_i is not a ray of the cone σ. For such a point $P \in \mathbb{A}^d(\mathcal{O}_v) \subset \mathcal{X}_\Sigma$, it follows that $n_v(\mathcal{D}_i^c, P) = v(p_i)$. Therefore, if we write $p_i = u_i\pi^{m_i}$ for units $u_1, \ldots, u_n \in \mathcal{O}_v^\times$ and $m_1, \ldots, m_n \in \mathbb{N}^*$, then $\phi_v(u_1\pi^{m_1} : \cdots : u_n\pi^{m_n}) = \sum_{i=1}^n m_i n_{\rho_i}$. We will now prove that this equality is still true even without the constraints on the p_i. By the equality

$$\mathcal{G}(K_v) = \{(t_1, \ldots, t_n) \mid \prod_{i=1}^{n} t_i^{\langle e_j, n_{\rho_i}\rangle} = 1 \text{ for all } 1 \leq j \leq d\},$$

where $e_1, \ldots, e_d$ is a choice of a basis of $N^\vee$, we see that $(\pi^{m_1} : \cdots : \pi^{m_n}) = (1 : \cdots : 1)$ if and only if $\sum_{i=1}^n m_i n_{\rho_i} = 0$. Therefore if $(\pi^{m_1} : \cdots : \pi^{m_n}) = (\pi^{m_1'} : \cdots : \pi^{m_n'})$ then $\sum_{i=1}^n (m_i - m_i') n_{\rho_i} = 0$ so $\sum_{i=1}^n m_i n_{\rho_i} = \sum_{i=1}^n m_i' n_{\rho_i}$.

Thus we see that

$$\phi_v(u_1\pi^{m_1} : \cdots : u_n\pi^{m_n}) = \sum_{i=1}^{n} m_i n_{\rho_i},$$

for all $m_i \in \mathbb{Z}$ and units $u_i \in \mathcal{O}_v^\times$. In particular it is clear that ϕ_v is a group homomorphism with kernel $\mathcal{U}(\mathcal{O}_v)$.

The surjectivity follows from the fact that $n_{\rho_1}, \dots, n_{\rho_n}$ span N as a lattice, combined with the identity $\phi_v(1 : \cdots : 1 : \pi : 1 : \cdots : 1) = n_{\rho_i}$, where i is index of the coordinate different from 1. The splitting is a direct consequence of the fact that N is a free abelian group.

For verifying the identity $\phi_v \circ f = \overline{f} \circ \phi_v$, it suffices to consider affine opens $\mathbb{A}^{d'}_{\mathcal{O}_v} \subset \widetilde{\mathcal{X}}$ and $\mathbb{A}^{d}_{\mathcal{O}_v} \subset \mathcal{X}$ such that f restricts to a morphism $\mathbb{A}^{d'}_{\mathcal{O}_v} \to \mathbb{A}^{d}_{\mathcal{O}_v}$. Now by comparing this map with the map $\overline{f}$, the result directly follows.

The final claim follows from the previous part by noticing that $\overline{f}$ is just the identity. □

Remark 3.2.5. By Proposition 3.2.4, the description of $\mathcal{M}$-points on projective space as in Section 2.1.4 generalizes to toric pairs (X, M), with X complete and smooth, with toric integral model $(\mathcal{X}, \mathcal{M})$. By replacing the homogeneous coordinates from that section with Cox coordinates, we obtain a description for the $\mathcal{M}$-points on $(\mathcal{X}, \mathcal{M})$.

Now we can extend the definitions of N_M and N_M^+ to the singular case.

Definition 3.2.6. Let X be a complete normal split toric variety with lattice of cocharacters N. We define $N_M = N_{f^*M}$ and $N_M^+ = N_{f^*M}^+$ for any toric birational morphism $f\colon Y \to X$, where Y is a complete smooth split toric variety, such that $D_i \times_X Y$ is a Cartier divisor for all $i \in \{1, \dots, n\}$.

Such a Y can always be found by taking $g\colon Z \to X$ to be the successive blowing up of the D_i, so that $g^{-1}D_i$ is a Cartier divisor, and then taking Y to be a toric resolution of singularities of Z.

It follows from Proposition 3.2.4 that N_M and N_M^+ are independent of the choice of the morphism f, so they are well-defined. This is because for any two resolutions of singularities of X, there exists a common refinement of both.

Remark 3.2.7. If X is a normal split toric variety such that $\mathrm{Cl}(X)$ contains torsion, then $N_M^+ = N$ for the trival pair $(X, M) = (X, 0)$. On the other hand, Proposition 3.1.7 implies that the ray generators n_{ρ_i} do not generate N, even as a group. Therefore, it is sometimes necessary to consider a resolution of singularities rather than directly trying to apply Definition 3.2.2, as that can give monoids which are too small.

The next notion measures divisibility of the unit group of completions of the field K.

Definition 3.2.8. For a PF field (K, C) we define $\rho(K, C)$ to be the set of $n \in \mathbb{N}^*$ such that the group $\mathcal{O}_v^\times$ is n-divisible for all $v \in \Omega_K$.

The set $\rho(K, C)$ is a submonoid of $\mathbb{N}^*$ generated by a subset of the prime numbers. In order to describe this notion for function fields we introduce the following definitions:

Definition 3.2.9. Let k be a field and let $n > 1$ be an integer with $\mathrm{char}(k) \nmid n$. We say that k is *n-closed* if one of the following equivalent properties hold:

1. For every finite extension l/k, the degree $[l : k]$ is coprime to n.

2. For every finite extension l/k, the group $l^\times$ is n-divisible.

Example 3.2.10. Separably closed fields are n-closed for all n not divided by the characteristic. For any prime number p and an integer $n > 1$ with $p \nmid n$ the union of finite fields $\bigcup_{m\geq 1} \mathbb{F}_{p^{n^m}}$ is n-divisible. Similarly for a separably closed field k with $\mathrm{char}(k) = p$ and an integer $n > 1$ with $p \nmid n$, the field $\bigcup_{m\geq 1} k(\!(t^{-1/n^m})\!)$ is n-divisible.

Recall that a field k is formally real if there exists an ordering on k and it is formally Euclidean if this ordering can be chosen such that every nonnegative element is a square.

Definition 3.2.11. We say that k is *hereditarily Euclidean* if every formally real algebraic extension of k is formally Euclidean.

The following lemma allows us to easily compute $\rho(K, C)$ for both number fields and function fields.

Lemma 3.2.12. *For any PF field (K, C), the monoid $\rho(K, C)$ is computed as follows:*

1. *If K is a number field then $\rho(K, C) = 1$.*
2. *If k is a field and $K = k(C)$, where C is a regular curve over k, then a prime number p belongs to $\rho(K, C)$ if and only if either*
   (a) *$p \neq \mathrm{char}(k)$ and k is p-closed,*
   (b) *or all of the following are satisfied:*
      - *k is a hereditarily Euclidean field,*
      - *$p = 2$,*
      - *$C(k') = \emptyset$, where k' is a real closure of k.*

Furthermore, for any PF field (K, C), if $n \in \mathbb{N}^*$ is an integer such that $n \notin \rho(K, C)$, then there are infinitely many places $v \in \Omega_K$ such that $\mathcal{O}_v^\times$ is not n-divisible.

Remark 3.2.13. In particular, if $K = k(C)$ with k a finite field, a number field or a function field (of transcendence degree at least 1), then $\rho(K, C) = 1$. If on the other hand, k is separably closed, then $\rho(K, C) = \{n \in \mathbb{N}^* \mid \mathrm{char}(k) \nmid n\}$. Finally, if $k = \mathbb{R}$, then $\rho(K, C) = \{n \in \mathbb{N}^* \mid 2 \nmid n\}$ if $C(\mathbb{R}) \neq \emptyset$ and $\rho(K, C) = \mathbb{N}^*$ otherwise.

Proof. We will prove the last statement in tandem with the computation of $\rho(K, C)$. Note that for this statement we can assume that $n = p$ is a prime. We split up the proof in two cases, depending on whether K is a number field or a function field. We first treat the case where K is a number field. For every prime number p there exist infinitely many prime numbers $q \equiv 1 \bmod p$ by Dirichlet's theorem on arithmetic progressions. For each place $v \in \Omega_K$ above such a prime number q, the group of units of the residue field $k_v^\times$ is not p-divisible since the order of $k_v^\times$ is divisible by p. Thus $\mathcal{O}_v^\times$ is not p-divisible either for such v. In particular, $\mathcal{O}_v^\times$ is not p-divisible for infinitely many places $v \in \Omega_K$ and therefore $\rho(K, C) = 1$.

Now we treat the case where K is a function field of a curve over a ground field k. For any place $v \in \Omega_K$, the completion is given by $\mathcal{O}_v \cong k_v[\![t]\!]$, where k_v is the residue field at v. For any $f \in k_v[\![t]\!]^\times$ we can write $f = ag$ where $a \in k_v^\times$ and $g \in k_v[\![t]\!]^\times$

has constant coefficient 1. Therefore f is a p-th power if and only if a and g are p-th powers. If $p \neq \mathrm{char}(k)$, then for any $x \in k_v[\![t]\!]$ with $|x|_v < 1$ the p-th root $\sqrt[p]{1+x}$ is well-defined and lies in $k_v[\![t]\!]$. In particular, since $|g-1|_v < 1$, g is a p-th power as long as $p \neq \mathrm{char}(k)$. If on the other hand $p = \mathrm{char}(k)$, then $1+t \in k_v[\![t]\!]^\times$ is not a p-th power. Therefore $\mathcal{O}_v^\times$ is p-divisible if and only if $p \neq \mathrm{char}(k)$ and $k_v^\times$ is p-divisible.

Therefore it follows that $p \in \rho(K, C)$ if k is p-closed and $p \neq \mathrm{char}(k)$ since in that case k_v is p-closed for all $v \in \Omega_K$. Similarly, $2 \in \rho(K, C)$ if k is hereditarily Euclidean and $C(k') = \emptyset$ for a real closure k'/k, since then k_v is 2-closed for all $v \in \Omega_K$.

For the other direction, we assume that $\mathcal{O}_v^\times$ is p-divisible for all but finitely many places. We will show p satisfies the conditions given in the statement of the lemma, and thus in particular $p \in \rho(K, C)$.

As we have seen we must have that $p \neq \mathrm{char}(k)$ and that for all but finitely many places $v \in \Omega_K$, the group $k_v^\times$ is p-divisible. First we prove that for any field $\tilde{k}$, if $\tilde{k}$ is not p-divisible, but $l^\times$ is p-divisible for some some finite extension $l/\tilde{k}$, then $p = 2$ and $\tilde{k}$ is an Euclidean field. If $p = 2$, then this follows from [EW87, Lemma 2(2)]. Assume therefore that p is odd. Let $a \in \tilde{k}$ be an element which is not a p-th power. Then [Lan02, Theorem 9.1] shows that the polynomial $X^{p^n} - a$ is irreducible over $\tilde{k}$ for every $n \geq 1$. Thus if $l/\tilde{k}$ is a finite extension, then $X^{p^n} - a$ does not have a linear factor over l if p^n is larger than the degree of the extension $l/\tilde{k}$. Therefore $l^\times$ is not p-divisible either since a is not a p^n-th power in l.

Since for any place $v \in \Omega_K$, k_v/k is a finite extension, $k^\times$ has to be either p-divisible or $p = 2$ and k is Euclidean.

Now assume that $k^\times$ is p-divisible and that $k_v^\times$ is p-divisible for all but finitely many places $v \in \Omega_K$, as before. We will show that this implies that k is p-closed. Since $k^\times$ is p-divisible, k is not Euclidean if $p = 2$. If k is not p-closed, then there exists a finite extension l/k such that $l^\times$ is not p-divisible. We can factor l/k as a separable extension $\tilde{l}/k$ followed by a totally inseparable extension $l/\tilde{l}$ by [Stacks, Tag 030K]. Since any separable extension of k is simple, $\tilde{l}$ is contained in the residue field of infinitely many closed points in $\mathbb{P}^1_k$. Thus, since there exists a dominant morphism $C \to \mathbb{P}^1_k$, there exist infinitely many places v of K for which $\tilde{l} \subset k_v$. This implies that $k_v^\times$ is not p-divisible for infinitely many places $v \in \Omega_K$, which is a contradiction. Thus if $k^\times$ is p-divisible, then $\tilde{l}^\times$ is p-divisible for every separable extension $\tilde{l}/k$.

For any totally inseparable extension $l/\tilde{l}$, any element $\alpha \in l^\times$ has a minimal polynomial of the form $X^{q^n} - \alpha^{q^n}$ for some $n \in \mathbb{N}$, where $q = \mathrm{char}(k)$ and $\alpha^{q^n} \in \tilde{l}^\times$. Since α^{q^n} is a p-th power in $\tilde{l}$, α is a p-th power in l and thus $l^\times$ is also p-divisible. Therefore, we see that k is p-closed. By the same argumentation, we also see that if k is Euclidean, then k is hereditarily Euclidean.

If k is hereditarily Euclidean, but C contains a k'-rational point for its real closure k', then $C(k')$ is infinite since a real closed field is an ample field [Pop96]. Therefore there are infinitely many places v such that $k_v^\times$ is not 2-divisible. □

The next lemma will be used in the proof of Theorem 1.1.3 over function fields.

Lemma 3.2.14. *Let k be a field and let C be a projective regular curve over k and let $p \in \rho(K, C)$ be a prime number. Then for any affine open $B \subset C$,* $\mathrm{Pic}(B)$ *is a p-divisible group.*

Proof. We first assume that k is p-closed. Since the statement only depends on p and on the scheme B, we can as in Remark 1.3.3 assume without loss of generality that k is algebraically closed in K, so that B is a geometrically integral curve over k. The connected component of the identity of the Picard scheme $J = \mathrm{Jac}(C)$ is a group scheme of finite type over k [BLR90, Section 8.2, Theorem 3]. We will first prove that $J(k)$ is p-divisible. The multiplication-by-p map $[p]$ gives the exact sequence of G-modules

$$0 \to J(k^{\mathrm{sep}})[p] \to J(k^{\mathrm{sep}}) \xrightarrow{[p]} J(k^{\mathrm{sep}}) \to 0,$$

where k^{sep} is the separable closure of k and $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ is the absolute Galois group of k. By the induced long exact sequence in Galois cohomology, this implies that $J(k)$ is p-divisible if $H^1(G, J(k^{\mathrm{sep}})[p]) = 0$, which we will now show.

By [Liu06, Chapter 7, Corollary 5.23.], $J(k^{\mathrm{sep}})[p]$ is finite since p is different from the characteristic. Furthermore every finite quotient of G has order coprime to p, since k is p-closed. Thus by [Har20, Corollary 1.49] we see that, for any normal subgroup $H \lhd G$ of finite index, $H^1(G/H, (J(k^{\mathrm{sep}})[p])^H) = 0$. Thus the inflation-restriction exact sequence [Har20, Theorem 1.42] implies that the restriction map $H^1(G, J(k^{\mathrm{sep}})[p]) \to H^1(H, J(k^{\mathrm{sep}})[p])^{G/H}$ is injective. For every continuous map $G \to J(k^{\mathrm{sep}})[p]$, one of the fibers contains an open neighbourhood of $1 \in G$, and therefore a normal subgroup H of finite index. Thus the map of the associated cohomology class gets sent to 0 by the restriction map. Since the map is injective, $H^1(G, J(k^{\mathrm{sep}})[p]) = 0$ and thus we see that $J(k)$ is p-divisible.

By applying [BLR90, Section 8.1, Proposition 4] with $S = T = \operatorname{Spec} k$, we have an exact sequence

$$0 \to \mathrm{Pic}^0(C) \to J(k) \to \mathrm{Br}(k),$$

where $\mathrm{Pic}^0(C)$ is the group of isomorphism classes of invertible sheaves on C with degree 0 and $\mathrm{Br}(k)$ is the Brauer group of k. Since k is p-closed, $\mathrm{Br}(k)[p] = 0$, since there does not exist a central division algebra of degree p over k, as its splitting field would have degree p over k. Thus it follows that $\mathrm{Pic}^0(C)$ is also p-divisible.

Since C is geometrically connected, the degree map gives an isomorphism $\mathrm{Pic}(C)/\mathrm{Pic}^0(C) \cong \mathbb{Z}$. Since the restriction map $\mathrm{Pic}(C) \to \mathrm{Pic}(B)$ is surjective, there is an exact sequence

$$\mathrm{Pic}^0(C) \to \mathrm{Pic}(B) \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

Here m is the greatest common divisor of the degrees of closed points in $C \setminus B$. The degree of a closed point on C is not divisible by p, as the residue field of such a divisor would be an extension of k with degree divisible by p. Therefore p does not divide m, so $\mathbb{Z}/p\mathbb{Z}$ is p-divisible and thus $\mathrm{Pic}(B)$ is p-divisible.

Now we treat the case where k is not p-closed. As we have seen in Lemma 3.2.12, this implies that $p = 2$, k is hereditarily Euclidean and C is a curve with no points defined over a real closure of k. This implies that every divisor on C defined over k is of the form $D = D' + \sigma(D')$, where D' is a divisor defined over the unique quadratic extension $l = k(\sqrt{-1})$ of k, and σ is the automorphism of l/k that sends $\sqrt{-1}$ to $-\sqrt{-1}$. As l is 2-closed by [EW87, Lemma 2], it follows by the previous argument that $\mathrm{Pic}(B_l)$ is 2-divisible, so there exists a divisor D'' over l such that $2D'' \sim D'$, and thus we see $D \sim 2(D'' + \sigma(D''))$. Therefore $\mathrm{Pic}(B)$ is 2-divisible as well. □

3.2.2 Squarefree strong approximation on the affine line

In the proof of Theorem 1.1.3, we will need the following definitions.

Definition 3.2.15. Let R be an Dedekind domain. An element $r \in R$ is *squarefree* if r is not contained in $\mathfrak{p}^2$ for any prime ideal $\mathfrak{p} \subset R$. If $R = \mathcal{O}(B)$ for some regular curve B over a field k, then we call an element $r \in R$ *separable* if it is squarefree in $R \otimes_k k'$ for every extension k' of k.

Note that if $R = k[t] = \mathcal{O}(\mathbb{A}^1_k)$, then we recover the familiar notions of squarefree polynomials and separable polynomials. If k is perfect, then $r \in \mathcal{O}(B)$ is separable if and only if it is squarefree.

Before we can carry out the proof of Theorem 1.1.3, we prove a stronger version of strong approximation on $\mathbb{A}^1$ for both global fields and function fields.

Lemma 3.2.16. *Let (K, C) be a global field, let $S \subset \Omega_K$ be a finite set of places containing a distinguished place $v_0 \in S$. Let $x_v \in K_v^\times$ for $v \in S$, and $\epsilon > 0$. Then there exist infinitely many pairwise coprime squarefree elements $f \in \mathcal{O}(B)$ such that*

$$|f - x_v|_v < \epsilon \text{ for all } v \in S \setminus \{v_0\},$$

where $B = C \setminus S$.

Let T be an integer and assume that v_0 is an infinite place if K is a number field. If $|x_{v_0}|_{v_0}$ is sufficiently large, depending on ϵ, T and $|x_v|_v$ for $v \in S \setminus \{v_0\}$, then there exist at least T such f which additionally satisfy

$$|f - x_{v_0}|_{v_0} < \epsilon |x_{v_0}|_{v_0}.$$

Furthermore, f can be taken to be a prime element if K is a function field and v_0 is a k-rational point, or if K is a number field (with no condition on v_0). In general, f can be taken to be the product of two prime elements.

Remark 3.2.17. Note that for $K = \mathbb{Q}$ this lemma is just a consequence of the prime number theorem for arithmetic progressions [BMO$^+$18, Theorem 1.1]. If K is a number field and if S does not contain an infinite place, then the statement from the lemma follows from Chebotarev's density theorem applied to L/K, where L is the ray class field associated to the modulus ∞I (see [Cox22, Chapter 2, §8]), where ∞ is the product of the infinite places and I is the ideal in $\mathcal{O}_K$ consisting of the elements $x \in \mathcal{O}_K$ with $|x|_v < \epsilon$ for all $v \in S$.

Remark 3.2.18. If S contains all infinite places in the setting of Lemma 3.2.16, then there exist only finitely many coprime elements $f \in \mathcal{O}(B)$ satisfying $|f - x_v|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$ and $|f - x_{v_0}|_{v_0} < \epsilon |x_{v_0}|_{v_0}$, since these inequalities imply an upper bound on the norm of the ideal $(f) \in B$.

Proof. The proof of the lemma uses the language of ideles and mainly relies on [Lan94, Chapter XV, Theorem 6], which is an equidistribution result that can be viewed as a generalization of Chebotarev's density theorem. For more background on ideles and equidistribution, see [Lan94, Chapter VII] and [Lan94, Chapter XV], respectively.

Denote the idele group of K by $\mathbf{J} = \prod_{v \in \Omega_K} (K_v^\times, \mathcal{O}_v^\times)$ and denote the S-idele group by $\mathbf{J}_S = \prod_{v \in \Omega_K \setminus S} \mathcal{O}_v^\times \times \prod_{v \in S} K_v^\times$. The norm of an idele $a \in \mathbf{J}$ is $\|a\| = \prod_{v \in \Omega_K} |a_v|_v$. We denote the subgroups of elements of norm 1 in $\mathbf{J}$ and $\mathbf{J}_S$ by $\mathbf{J}^0$ and $\mathbf{J}_S^0$.

The proof is split up in two parts, depending on whether K is a number field or a function field. While the proofs of these cases differ, they follow the same general ideas and both hinge on applying [Lan94, Chapter XV, Theorem 6] to a retraction $\phi\colon \mathbf{J} \to \mathbf{J}^0$. First we give some generalities common to both proofs.

Step 0. Without loss of generality we can assume that $\epsilon < \min_{v\in S\setminus\{v_0\}}(1, |x_v|_v)$. Choose for each finite place $\mathfrak{q} \in \Omega_K^{<\infty}$ a uniformizer $\pi_\mathfrak{q} \in K^\times$ at $\mathfrak{q}$. Define the map $\tau\colon \Omega_K^{<\infty} \to \mathbf{J}$ by $(\tau(\mathfrak{q}))_v = \pi_\mathfrak{q}$ if $v \neq \mathfrak{q}$ and $(\tau(\mathfrak{q}))_\mathfrak{q} = 1$. Note that for $\mathfrak{q} \notin S$, $\tau(\mathfrak{q}) \in K^\times \mathbf{J}_S$ if and only if $\mathfrak{q}$ is a principal ideal in $\mathcal{O}(B)$. This is because $\tau(\mathfrak{p}) \in K^\times \mathbf{J}_S$ means that there exists $u \in K^\times$ such that $u\pi_\mathfrak{q} \in \mathcal{O}_v^\times$ for all $v \in \Omega_K \setminus (S \cup \{\mathfrak{q}\})$ and such that $u \in \mathcal{O}_\mathfrak{q}^\times$, so $(u\pi_\mathfrak{q}) \in B$ is a prime ideal.

Step 1 for number fields: v_0 infinite. We assume without loss of generality that S contains all infinite places. We also first treat the case when v_0 is an infinite place. Define the retraction $\phi\colon \mathbf{J} \to \mathbf{J}^0$ by

$$\phi(a)_v = \begin{cases} a_v & \text{if } v \neq v_0, \\ a_v/\|a\|^{1/e} & \text{if } v = v_0, \end{cases}$$

where $e = 1$ if v_0 is real and $e = 2$ if v_0 is complex. We define σ to be the composition of ϕ with the quotient map $\mathbf{J}^0 \to \mathbf{J}^0/K^\times$. Then $\sigma(\mathbf{J}^0) = \mathbf{J}^0/K^\times$, and $\sigma(K^\times) = 1$. Therefore, by [Lan94, Chapter XV, Theorem 6], $\Omega_K^{<\infty}$ is λ-equidistributed in $\mathbf{J}^0/K^\times$, where $\lambda = \sigma \circ \tau$. The map τ here is defined differently from [Lan94], but the composition yields the same map λ after composing with the inversion map $(\cdot)^{-1}\colon \mathbf{J}^0/K^\times \to \mathbf{J}^0/K^\times$ and therefore the conclusion still follows.

Let

$$\begin{aligned} U :=& \left\{ z \in \mathbf{J}_S^0 \colon |z_v - x_v|_v < \epsilon \min\left(1, \frac{|x_v|_v}{2^{r+2}}\right), \forall v \in S \setminus \{v_0\} \right\} \\ &\cap \left\{ z \in \mathbf{J}_S^0 \colon |\arg(z_{v_0}) - \arg(x_{v_0})|_{v_0} < \frac{\epsilon}{4} \right\}, \end{aligned}$$

be an open set of $\mathbf{J}^0$, where r is the cardinality of S and $\arg(z)$ is the principal argument of $z \in \mathbb{C}^\times$. Note that U is a nonempty open set of $\mathbf{J}^0$. Denote the image of U in $\mathbf{J}^0/K^\times$ by $\overline{U}$. The maps we have defined together form the following commutative diagram

$$\begin{tikzcd} & & U \arrow[r] \arrow[d, hook] & \overline{U} \arrow[d, hook] \\ \Omega_K^{<\infty} \arrow[r, "\tau"] \arrow[rrr, bend right, "\lambda"'] & \mathbf{J} \arrow[r, "\phi"] \arrow[rr, bend right, "\sigma"'] & \mathbf{J}^0 \arrow[r] & \mathbf{J}^0/K^\times. \end{tikzcd}$$

As the indicator function of $\overline{U}$ is integrable, a positive density of prime ideals $\mathfrak{q} \in B$, ordered by their norms, satisfy $\lambda(\mathfrak{q}) \in \overline{U}$, by the definition of equidistribution given in [Lan94, page 316]. Since the image of $\mathbf{J}_S^0$ in $\mathbf{J}^0/K^\times$ is $K^\times\mathbf{J}_S^0/K^\times$, such prime ideals are principal as we have seen in Step 0. By the Landau prime ideal theorem [Lan03, page 670] the number of prime ideals of norm up to X grows asymptotically as $X/\log X$. Therefore the number of prime ideals $\mathfrak{q} \in B$ with $\lambda(\mathfrak{q}) \in \overline{U}$ of norm up to X grows asymptotically as $cX/\log X$ for some constant $c > 0$. Therefore, for $N_0 \in \mathbb{R}$ sufficiently large, there exist T distinct principal prime ideals $\mathfrak{q} = (q) \in B$ with $\lambda((q)) \in \overline{U}$ and of norm $(1-\frac{\epsilon}{4})N_0 < N(q) < (1+\frac{\epsilon}{4})N_0$. Note that $(\phi\circ\tau)(q)_v = uq$ for all $v \in \Omega_K \setminus \{v_0, \mathfrak{q}\}$, $(\phi \circ \tau)(q)_{\mathfrak{q}} = u$ and $(\phi \circ \tau)(q)_{v_0} = uq/N(q)^{1/e}$, where $u \in K^\times \cap \mathcal{O}_{\mathfrak{q}}^\times$ and $N(q) = \prod_{v \in S} |q|_v$ is the norm of q in $\mathcal{O}(B)$.

By definition of $\overline{U}$, for all prime ideals (q) of $\mathcal{O}(B)$ with $\lambda((q)) \in \overline{U}$, there exists $u' \in K^\times$ such that $u(\phi \circ \tau)(q) \in U$. This implies that $uu'q \in \mathcal{O}_v^\times$ for all $v \in \Omega_K \setminus \{v_0, \mathfrak{q}\}$, $uu' \in \mathcal{O}_{\mathfrak{q}}^\times$ and $|\arg((uq/N(q)^{1/e})_{v_0}) - \arg(x_{v_0})|_{v_0} < \frac{\epsilon}{4}$. Thus $uu'q \in \mathcal{O}(B)$, $(uu'q) = (q)$ and $uu'q \in U$ via the natural embedding $K^\times \subset \mathbf{J}^0$, since the argument of $uu'q$ in K_{v_0} is not affected by scaling by a positive real number. Therefore, we can choose the prime element $q \in \mathcal{O}(B)$ to lie in U itself.

Without loss of generality, we assume $|x_{v_0}|_{v_0}$ is sufficiently large so that $N_0 = \prod_{v \in S} |x_v|_v$ is large enough for T pairwise coprime elements $q \in U$ to exist. Then we have

$$(1 - \tfrac{\epsilon}{4})|x_{v_0}|_{v_0} < \frac{N(q)}{\prod_{v \in S \setminus \{v_0\}} |x_v|_v} < (1 + \tfrac{\epsilon}{4})|x_{v_0}|_{v_0}, \tag{3.2.3}$$

for such $p \in U$.

The triangle inequality implies $\left|\frac{q}{x_v}\right|_v - 1 \leq \left|\frac{q}{x_v} - 1\right|_v \leq \left|\frac{q}{x_v}\right|_v + 1$ and by combining this with the definition of U we find $1 - \frac{\epsilon}{2^{r+2}} < \frac{|q|_v}{|x_v|_v} < 1 + \frac{\epsilon}{2^{r+2}}$ for $v \in S \setminus \{v_0\}$, and thus

$$1 - \tfrac{\epsilon}{8} < \prod_{v \in S \setminus \{v_0\}} \frac{|p|_v}{|x_v|_v} < 1 + \tfrac{\epsilon}{8}. \tag{3.2.4}$$

By combining the inequalities (3.2.3) and (3.2.4), we obtain

$$\left(1 - \frac{\epsilon}{2}\right) |x_{v_0}|_{v_0} < |q|_{v_0} < \left(1 + \frac{\epsilon}{2}\right) |x_{v_0}|_{v_0}.$$

If v_0 is real, then the inequalities on the argument show that $v_0(q)$ has the same sign as x_{v_0} so $\left|\frac{q}{x_{v_0}} - 1\right|_{v_0} < \frac{\epsilon}{2}$. If v_0 is complex, then the inequalities on the argument show

$$\left|\frac{q}{x_{v_0}} - 1\right|_{v_0} < \left|\left(1 + \frac{\epsilon}{2}\right) e^{\epsilon i/4} - 1\right|^2 < \left(\frac{\epsilon}{2} + \left(1 + \frac{\epsilon}{2}\right)\frac{\epsilon}{4} + \sum_{n=2}^{\infty} \frac{(\epsilon/4)^n}{n!}\right)^2 < \epsilon^2 < \epsilon.$$

This proves the existence of T coprime prime elements $f = q \in U$ satisfying the second condition of the lemma and by definition of U, they also satisfy the first condition.

Step 2 for number fields: v_0 finite. Now we will prove the lemma when K is a number field and v_0 is a finite place. We will derive this from the previously treated case when v_0 was infinite by choosing an infinite place $v' \in S$ and by letting it play the role of v_0 so that we can apply the previously proven case of the lemma. By the generalization of Dirichlet's unit theorem to S-integers [Nar04, Theorem 3.12], there exists $u \in \mathcal{O}(B)^\times$ with $|u|_v = 1$ for all finite places $v \in \Omega_K^{<\infty} \setminus \{v_0\}$ and positive valuation at v_0. Therefore by the product formula there exists an infinite place v' such that $|u|_{v'} > 1$, and by taking powers of we can take $|u|_{v'}$ to be larger than any given bound.

For $\epsilon' > 0$ and any integer $R > 0$, if $|u|_{v'}$ is sufficiently large, the part of the lemma proven in Step 1 implies that we can find R pairwise coprime prime elements $q \in \mathcal{O}_S$ such that $|q - ux_v|_v < \epsilon'$ for all places $v \in S \setminus \{v_0, v'\}$ and $|q - ux_{v'}|_{v'} < \epsilon'|ux_{v'}|_{v'}$. If we set $f = q/u$, then $|f - x_v|_v < \epsilon'$ for all places $v \in S \setminus \{v_0, v'\}$ and $|f - x_{v'}|_{v'} < \epsilon'|x_{v'}|_{v'}$.

As T was arbitrary, this implies that for every $\epsilon > 0$ there exist infinitely many pairwise coprime prime elements $f \in \mathcal{O}(B)$ such that $|f - x_v|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$ since we can take $\epsilon' = \epsilon / \max(1, |x_{v'}|_{v'})$.

Step 1 for function fields: v_0 a rational point. Now we will prove the statement for global function fields using a similar strategy as for number fields, relying on [Lan94, Chapter XV, Theorem 6]. While this theorem is formulated for number fields, the statement is true for global fields. This is because the proof of this result relies on Theorems 1, 2, 3 and 5 as well as Proposition 1 in [Lan94, Chapter XV]. Theorem 1 and Proposition 1 are purely analytic statements, not involving number fields, while Theorem 2, 3 and 5 are true over global fields using the same argumentation as given in the book. As noted in Remark 1.3.3, we can take k to be the field of constants of C so that C is geometrically integral.

We will first assume that v_0 is a k-rational point and prove the general case afterwards. Let l be the cardinality of k and define the retraction $\phi\colon \mathbf{J} \to \mathbf{J}^0$ by

$$\phi(a)_v = \begin{cases} a_v & \text{if } v \neq v_0, \\ a_v / \pi_{v_0}^{\log_l \|a\|} & \text{if } v = v_0, \end{cases}$$

where $\pi_{v_0} \in \mathcal{O}_{v_0}$ is a uniformizer and $\|a\|$ is the norm of a. We define σ to be the composition of ϕ with the quotient map $\mathbf{J}^0 \to \mathbf{J}^0/K^\times$, as in the case for number fields. Then $\sigma(\mathbf{J}^0) = \mathbf{J}^0/K^\times$, and $\sigma(K^\times) = 1$.

Therefore we can use [Lan94, Chapter XV, Theorem 6] as in Step 1 for number fields to conclude that Ω_K is λ-equidistributed in $\mathbf{J}^0/K^\times$, where $\lambda = \sigma \circ \tau$.

For any $b \in K_{v_0}^\times$ satisfying $|b|_{v_0} \prod_{v \in S\setminus\{v_0\}} |x_v|_v = 1$, define the nonempty open subset

$$U_b := \left\{ z \in \mathbf{J}_S^0 \colon \begin{array}{ll} |z_v - x_v|_v & < \epsilon, \forall v \in S \setminus \{v_0\} \\ |z_{v_0} - b|_{v_0} & < \epsilon / \prod_{v \in S\setminus\{v_0\}} |x_v|_v \end{array} \right\}$$

and denote its image in $\mathbf{J}^0/K^\times$ by $\overline{U}_b$. As in the proof for number fields, the maps defined fit into the following commutative diagram

$$\begin{tikzcd} & & U_b \arrow[r] \arrow[d, hook] & \overline{U}_b \arrow[d, hook] \\ \Omega_K^{<\infty} \arrow[r, "\tau"] \arrow[rrr, bend right, "\lambda"'] & \mathbf{J} \arrow[r, "\phi"] \arrow[rr, bend right, "\sigma"'] & \mathbf{J}^0 \arrow[r] & \mathbf{J}^0/K^\times. \end{tikzcd}$$

Note that $(\phi\circ\tau)(q)_v = uq$ for all $v \in \Omega_K \setminus \{v_0, \mathfrak{q}\}$, $(\phi\circ\tau)(q)_{\mathfrak{q}} = u$ and $(\phi\circ\tau)(q)_{v_0} = uq/\pi_{v_0}^{\log_l N(q)}$, where $u \in K^\times \cap \mathcal{O}_{\mathfrak{q}}^\times$ and $N(q) = \prod_{v \in S} |q|_v$ is the norm of q in $\mathcal{O}(B)$.

Let $n \geq 1$ be an integer. By the Hasse-Weil bound [Poo17, Corollary 7.2.1] for B over $\mathbb{F}_{l^n}$, $\mathcal{O}(B)$ has at least $l^n + O(l^{n/2})$ prime ideals of norm l^n, where implied constant depends on C but not on n. Thus if n is sufficiently large, then for every $b \in K_{v_0}^\times$ with $|b|_{v_0} \prod_{v \in S\setminus\{v_0\}} |x_v|_v = 1$ there exist at least T pairwise coprime primes $\mathfrak{q} = (q)$ of norm l^n with $\lambda(\mathfrak{q}) \in \overline{U}_b$. In particular, if $|x_{v_0}|_{v_0}$ is sufficiently large, then there exists at least T pairwise coprime primes $\mathfrak{q} = (q)$ of norm $\prod_{v \in S} |x_v|_v$ with $\lambda(\mathfrak{q}) \in \overline{U}_b$.

By definition of $\overline{U}_b$, for all prime ideals (q) of $\mathcal{O}(B)$ with $\lambda((q)) \in \overline{U}_b$, there exists $u' \in K^\times$ such that $u(\phi \circ \tau)(q) \in U_b$. This implies that for all $uu'q \in \mathcal{O}_v^\times$ for all $v \in \Omega_K \setminus \{v_0, \mathfrak{q}\}$, $uu' \in \mathcal{O}_{\mathfrak{q}}^\times$ and $|uu'q/\pi_{v_0}^{\log_l N(q)} - b|_{v_0} < \epsilon / \prod_{v \in S \setminus \{v_0\}} |x_v|_v$. This implies that $uu'q \in \mathcal{O}(B)$ and $(uu'q) = (q)$. Furthermore, if (q) has norm $N(q) = \prod_{v\in S} |x_v|_v$, this implies that $|uu'q - b\pi_{v_0}^{\log_l N(q)}|_{v_0} < \epsilon |x_{v_0}|_{v_0}$

In particular, by taking $b = x_{v_0}\pi_{v_0}^{-\log_l \prod_{v\in S}|x_v|_v}$ it follows that there exist T coprime prime elements $f = q \in \mathcal{O}_S$ with $|f - x_v|_v < \epsilon$ for $v \in S \setminus \{v_0\}$ and $|f - x_{v_0}|_{v_0} < \epsilon |x_{v_0}|_{v_0}$.

Step 2 for function fields: v_0 not a rational point. Now it remains to consider the case where v_0 is not a k-rational point. By the Hasse-Weil bound, there exists a place $v' \in \Omega_K \setminus S$ such that $\gcd(\deg(v), \deg(v')) = 1$ for every $v \in S$. For $v \in S$ choose a factorisation $x_v = c_v^{\deg(v_0)} d_v^{\deg(v')}$ where $c_v, d_v \in K_v^\times$ and such that $0 \le -v(d_v) < \deg(v_0)$ and thus $|d_v|_v \ge 1$. Let $\widetilde{k}$ be the splitting field of the closed point v' and let $\widetilde{K}$ be the fraction field of $C_{\widetilde{k}}$. Denote the complement of $S \sqcup \{v'\}$ in C by B'. For every place in S there is a unique place $\widetilde{v} \in \Omega_{\widetilde{K}}$ lying above it, by the coprimality assumption. Let $\widetilde{S}$ be the set of places in $\Omega_{\widetilde{K}}$ above the places in S. Every place $\widetilde{v}' \in \Omega_{\widetilde{K}}$ above v' has degree 1 and thus corresponds to a rational point on $B'_{\widetilde{k}}$. As we already know that the statement is true if v_0 is a rational point, we can apply the lemma to $C_{\widetilde{k}}$, where $B'_{\widetilde{k}}$ plays the role of B and $\widetilde{v}'$ plays the role of v_0, for some choice of $\widetilde{v}' \in \Omega_{\widetilde{K}}$ above v'. Therefore, for every $\epsilon_2 > 0$ we can find T coprime prime elements $\widetilde{q} \in \mathcal{O}(B'_{\widetilde{k}})$ such that for every place $v \in S$, we have

$$|\widetilde{q} - d_v|_{\widetilde{v}} < \epsilon_2.$$

By taking $\epsilon_2 < \min(|d_v|_{\widetilde{v}}, 1)$, we ensure that $|\widetilde{q}|_{\widetilde{v}} = |d_v|_{\widetilde{v}} = |d_v|_v^{\deg(v')}$.

By the Hasse-Weil bound $\mathcal{O}(B'_{\widetilde{k}})$ has $l^n + O(l^{n/2})$ prime ideals of norm l^n lying above completely split primes in $\mathcal{O}(B')$. This is because the Hasse-Weil bound implies that the number of primes ideals in $\mathcal{O}(B'_{\widetilde{k}})$ of norm l^n lying above primes in $\mathcal{O}(B')$ which are not completely split is bounded from above by $\sum_{d|n}\left(l^d + O(l^{d/2})\right)$. Therefore the prime element $\widetilde{q}$ can be chosen such that $q_2 := \widetilde{q}\sigma(\widetilde{q})\dots\sigma^{\deg(v')-1}(\widetilde{q})$ is a prime element in $\mathcal{O}(B')$, where σ is a generator of $\mathrm{Gal}(\widetilde{k}/k)$. Note furthermore that for all $v \in S$ and $a \in \widetilde{K}$, we have $|\sigma(a)|_{\widetilde{v}} = |a|_{\widetilde{v}}$, since $\widetilde{v}$ is the unique place above v. Therefore, by the ultrametric triangle inequality, there exist T coprime prime elements $q_2 \in \mathcal{O}(B')$ such that

$$\begin{aligned}
|q_2 - d_v^{\deg(v')}|_v^{\deg(v')} &= |q_2 - d_v^{\deg(v')}|_{\widetilde{v}} \\
&\le \max(|\widetilde{q}\sigma(\widetilde{q})\dots\sigma^{\deg(v')-2}(\widetilde{q})d_v - q_2|_{\widetilde{v}}, \dots, \\
&\qquad |d_v^{\deg(v')} - \widetilde{q}d_v^{\deg(v')-1}|_{\widetilde{v}}) \\
&< \epsilon_2 \prod_{i=0}^{\deg(v')-1} |\sigma^i(\widetilde{q})|_{\widetilde{v}} / \min(|\widetilde{q}|_{\widetilde{v}}, \dots, |\sigma^{\deg(v')-1}(\widetilde{q})|_{\widetilde{v}}) \\
&= \epsilon_2 |\widetilde{q}|_{\widetilde{v}}^{\deg(v')-1} = \epsilon_2 |q_2|_v^{\deg(v')-1}
\end{aligned}$$

for all $v \in S$.

By the same reasoning, for every $\epsilon_1 > 0$ and sufficiently large $|c_{v_0}|_{v_0}$, there also exist T pairwise coprime prime elements q_1, pairwise coprime to the chosen prime elements q_2, with

$$|q_1 - c_v^{\deg(v_0)}|_v^{\deg(v_0)} < \epsilon_1 |q_1|_v^{\deg(v_0)-1} \text{ for all } v \in S$$

and

$$|q_1 - c_{v_0}^{\deg(v_0)}|_{v_0}^{\deg(v_0)} < \epsilon_1 |q_1|_{v_0}^{\deg(v_0)},$$

and $|q_1 q_2|_{v'} = 1$. Hence $f := q_1 q_2$ is a squarefree element in $\mathcal{O}(B)$.

Note that for all $v \in S$ the ultrametric triangle inequality implies

$$|f - x_v|_v = |q_1 q_2 - c_v^{\deg(v_0)} d_v^{\deg(v')}|_v \leq \max(|q_2|_v |q_1 - c_v^{\deg(v_0)}|_v, |q_1|_v |q_2 - d_v^{\deg(v')}|_v).$$

Combining this inequality with the inequalities on $|q_1 - c_v^{\deg(v_0)}|_v$ and $|q_2 - d_v^{\deg(v')}|_v$ gives

$$\begin{aligned} |f - x_v|_v &\leq |x_v|_v \max\left(\epsilon_1 |q_1|_v^{-1/\deg(v_0)}, \epsilon_2 |q_2|_v^{-1/\deg(v')}\right) \\ &\leq |x_v|_v \max\left(\epsilon_1 |q_1|_v^{-1/\deg(v_0)}, \epsilon_2\right) \end{aligned}$$

for $v \in S \setminus \{v_0\}$ and

$$|f - x_{v_0}|_{v_0} \leq |x_{v_0}|_{v_0} \max\left(\epsilon_1, \epsilon_2 |q_2|_{v_0}^{-1/\deg(v')}\right) \leq |x_{v_0}|_{v_0} \max(\epsilon_1, \epsilon_2).$$

In particular if we choose

$$\epsilon_1 = \epsilon / \max_{v \in S \setminus \{v_0\}} (1, |x_v|_v |q_1|_v^{-1/\deg(v_0)})$$

and

$$\epsilon_2 = \epsilon / \max_{v \in S \setminus \{v_0\}} (1, |x_v|_v)$$

then f is a squarefree element in $\mathcal{O}(B)$ satisfying the desired conditions and by varying the choices for q_1 and q_2 there are at least T pairwise coprime elements f satisfying the conditions. □

Now we prove the analogous statement for function fields of a curve over an infinite field.

Lemma 3.2.19. *Let K be a function field of a regular projective curve C over an infinite field k and let $S \subset \Omega_K$ be a finite set of places containing a distinguished place $v_0 \in S$. For $v \in S$ Let $x_v \in K_v^\times$ and let $\epsilon > 0$. Then there exist infinitely many pairwise coprime separable elements $f \in \mathcal{O}(B)$ such that*

$$|f - x_v|_v < \epsilon \text{ for all } v \in S \setminus \{v_0\},$$

where $B = C \setminus S$. Furthermore, if $|x_{v_0}|_{v_0}$ is sufficiently large, depending on ϵ and $|x_v|_v$ for $v \in S \setminus \{v_0\}$, then there exist infinitely many such f which additionally satisfy

$$|f - x_{v_0}|_{v_0} < \epsilon |x_{v_0}|_{v_0}.$$

Proof. For $v \in S$ we write D_v for the divisor on C associated to the place v and $g(C)$ for the genus of C. Let $n > 0$ be some integer such that $p^{-n} < \epsilon$, where $p = \mathrm{char}(k)$ if k has positive characteristic and $p = 2$ if k has characteristic 0. For an integer $m > 0$ and a place $v \in S \setminus \{v_0\}$, we write

$$\tilde{D}_{v,m} = mD_{v_0} - v(x_v)D_v - \sum_{\tilde{v} \in S \setminus \{v_0, v\}} nD_{\tilde{v}}.$$

For every place $v \in S \setminus \{v_0\}$, Riemann-Roch [Liu06, Theorem 7.3.26] implies $h^0(\tilde{D}_{v,m}) = \deg(\tilde{D}_{v,m}) - g(C) + 1$ and $h^0(\tilde{D}_{v,m} - D_v) = \deg(\tilde{D}_{v,m}) - \deg(D_v) - g(C) + 1$, as long as $\deg(\tilde{D}_{v,m}) - \deg(D_v) > 2g(C) - 2$. This inequality is satisfied whenever m is large enough, so for such m we have

$$h^0(\tilde{D}_{v,m}) - h^0(\tilde{D}_{v,m} - D_v) \geq \deg(D_v) \geq 1.$$

Therefore there exists an element $\tilde{h}_v \in \mathcal{O}_C(\tilde{D}_{v,m}) \setminus \mathcal{O}_C(\tilde{D}_{v,m} - D_v) \subset \mathcal{O}(B)$, which therefore satisfies $v(\tilde{h}_v) = v(x_v)$ and $|\tilde{h}_v|_{\tilde{v}} < \epsilon$ for all $\tilde{v} \in S \setminus \{v_0, v\}$. Furthermore we have $|\tilde{h}_v|_{v_0} \leq p^{-m \deg(D_{v_0})} < \epsilon |x_{v_0}|_{v_0}$ whenever $|x_{v_0}|_{v_0}$ is sufficiently large. The divisor

$$\tilde{D}_{v_0} = -v_0(x_{v_0})D_{v_0} - \sum_{v \in S \setminus \{v_0\}} nD_v$$

is very ample if $|x_{v_0}|_{v_0}$ is sufficiently large, as this implies that $-v(x_{v_0})$ is a large positive integer, so in the same manner we construct $\tilde{h}_{v_0} \in \mathcal{O}(B)$ with $v_0(\tilde{h}_{v_0}) = v_0(x_{v_0})$ and $|\tilde{h}_{v_0}|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$.

For any $v \in S$ we have the inequality $v(x_v - c_v\tilde{h}_v) < v(x_v)$ for some $c_v \in k^\times$. For $v \in S \setminus \{v_0\}$, by applying the the above construction of $\tilde{h}_v$ to $x_v - c_v\tilde{h}_v$ instead of x_v we iteratively construct $\tilde{h}_{v,1}, \ldots, \tilde{h}_{v,r} \in \mathcal{O}(B)$ and $c_{v,1}, \ldots, c_{v,r} \in k^\times$ such that $|\tilde{h}_{v,1}|_{\tilde{v}}, \ldots, |\tilde{h}_{v,r}|_{\tilde{v}} < \epsilon$ for all $\tilde{v} \in S \setminus \{v_0, v\}$, $|\tilde{h}_{v,1}|_{v_0}, \ldots, |\tilde{h}_{v,r}|_{v_0} < \epsilon |x_{v_0}|_{v_0}$ and

$$v\left(x_v - \sum_{i=1}^{j} c_{v,i}\tilde{h}_{v,i}\right) < v\left(x_v - \sum_{i=1}^{j-1} c_{v,i}\tilde{h}_{v,i}\right)$$

for all $j \in \{1, \ldots, r\}$. By taking $r = v(x_v) + n$ and by setting

$$h_v = \sum_{i=1}^{r} c_{v,i}\tilde{h}_{v,i} \in \mathcal{O}(B),$$

the ultrametric triangle inequality implies $|h_v|_{\tilde{v}} < \epsilon$ for all $\tilde{v} \in S \setminus \{v, v_0\}$, $|h_v|_{v_0} < \epsilon |x_{v_0}|_{v_0}$ and $|h_v - x_v| < \epsilon$.

In this manner, we also construct $h_{v_0} \in \mathcal{O}(B)$ with $|h_{v_0}|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$ and $|h_{v_0} - x_{v_0}|_{v_0} < \epsilon |x_{v_0}|_{v_0}$. If we write $h = \sum_{v \in S} h_v$ then it follows that $|h - x_v|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$ and $|h - x_{v_0}| < \epsilon |x_{v_0}|_{v_0}$. So now we have found an $h \in \mathcal{O}(B)$ with the desired properties, except for the fact that h need not be separable. We resolve this by slightly perturbing h, by adding a function to it with small valuations at the places $v \in S$.

In a similar way as before we use Riemann-Roch to construct $g \in \mathcal{O}(B)$ satisfying $|g|_v < \epsilon$ for all $v \in S \setminus \{v_0\}$ and $|g|_{v_0} < \epsilon |x_{v_0}|_{v_0}$, whenever $|x_{v_0}|_{v_0}$ is sufficiently large, such that h and g do not share any zeroes. Additionally, we construct g such that $\deg(g)$ is not divisible by the characteristic of k and such that $\deg(g) > \deg(h)$. Then the closed subscheme

$$X = \{h + tg = 0\} \subset B \times_k \mathbb{A}^1_k$$

is integral, since h and g do not share any zeroes. The projection morphism $\pi\colon X \to \mathbb{A}^1_k$ has degree coprime to the characteristic and is therefore separable. By generic flatness [Stacks, Tag 052A], there exists an nonempty open $V \subset \mathbb{A}^1_k$ such that the restriction $X \times_{\mathbb{A}^1_k} V \to V$ of π is flat. Therefore, [Spr98, Proposition 2.4(1)] implies there exists a nonempty open $U \subset \mathbb{A}^1_k$ such that the fiber $\pi^{-1}(c)$ is geometrically regular for all $c \in U$. In particular, since k is infinite, there exist infinitely many $c \in k$ such that for $f = h + cg \in \mathcal{O}(B)$ the scheme $\operatorname{div}(f) \cap B$ is geometrically regular, which proves that f is separable. Furthermore, any two different choices of $c \in k$ yield functions f and $\tilde{f}$ which are coprime to each other. □

3.2.3 Proof of Theorem 1.1.3

In this section we prove Theorem 1.1.3, and thus completely characterize when a toric pair (X, M) satisfies M-approximation off a finite set of places T. We will treat the cases $T \neq \emptyset$ and $T = \emptyset$ separately.

By Corollary 3.2.1 we can assume without loss of generality that X is smooth. Furthermore, by Proposition 2.2.11 and Proposition 2.2.6 we can also assume and $M = M_{\text{fin}}$. We additionally assume without loss of generality that $\mathfrak{M} = \mathfrak{M}_{\text{red}}$.

The pair (X, M) satisfies M-appproximation off T if and only if for every finite set of places S containing $T \cup \Omega_K^\infty$, any choice of a point $Q_v = (q_{v,1} : \cdots : q_{v,n}) \in X(K_v)$ and any analytic open neighborhood $Q_v \in V_v$ for every $v \in S \setminus T$, there exists a rational point $Q = (q_1 : \cdots : q_n) \in X(K)$ such that $Q \in (\mathcal{X}, \mathcal{M})(B)$ and $Q \in V_v$ for every $v \in S \setminus T$. Here $B = \Omega_K \setminus S$ and $(\mathcal{X}, \mathcal{M})$ is the toric integral model of (X, M) over B. We will write d for the dimension of X, $\mathcal{U} \cong \mathbb{G}^d_{m,\mathbb{Z}}$ for the open torus in $\mathcal{X}$ and U for its base change to K.

Proof of sufficiency of the conditions. We will first show that (X, M) satisfies M-approximation off T if $T \neq \emptyset$ and $|N : N_M| \in \rho(K, C)$ or $T = \emptyset$ and $N = N_M^+$. The majority of the proofs of the two cases are the same, with only the last part of the proofs differing. The Cox morphism $\pi\colon \mathcal{Y} \to \mathcal{X}$, introduced in Section 3.1, induces for every $v \in S \setminus T$ a continuous map $\mathcal{Y}(K_v) \to \mathcal{X}(K_v)$. Therefore, there exists $\epsilon > 0$ such that if for any $Q = (q_1 : \cdots : q_n) \in X(K)$ is a point such that $|q_i - q_{v,i}|_v < \epsilon |q_{v,i}|_v$ for all $i \in \{1, \ldots, n\}$ and $v \in S \setminus T$, then $Q \in V_v$ for all $v \in S \setminus T$.

We will now show that we can reduce to the case where $\operatorname{mult}_v(Q_v) \in N_M$ for all $v \in S \setminus T$. If $\rho(K, C) = \{1\}$ or $T = \emptyset$, then this is trivially true since then $N_M = N$. In particular, we only need to show this when K is a function field.

If K is a function field, Lemma 3.2.14 implies that $\operatorname{Pic}(C \setminus T)$ is $|N : N_M|$-divisible. This means that for every divisor D on $C \setminus T$, there is $u \in K^\times$ such that $D + \operatorname{div} u = |N : N_M| D'$ for some divisor D' on $C \setminus T$. Therefore, there exists $\mathbf{u} = (u_1 : \cdots : u_n)$ with $u_1, \ldots, u_n \in K^\times$ such that $\operatorname{mult}_v(u_1 q_{v,1} : \cdots : u_n q_{v,n}) \in N_M$ for all $v \in S \setminus T$ and $\operatorname{mult}_v(\mathbf{u}) \in N_M$ for all $v \in \Omega_K^{<\infty} \setminus S$. Let S' be the finite set consisting of the places in S together with all places v for which $\operatorname{mult}_v(\mathbf{u}) \neq 0$. For

$v \in S \setminus T$, we set

$$Q'_v = (u_1 q_{v,1} : \cdots : u_n q_{v,n})$$

and $V'_v = \mathbf{u} V_v$, and set $Q'_v = \mathbf{u}$ and $V'_v = \mathbf{u}\mathcal{U}(\mathcal{O}_v)$ for $v \in S' \setminus S$, where the multiplication is done coordinate-wise. If there exists $Q' \in (\mathcal{X}, \mathcal{M})(B')$, where $B' = C \setminus S'$, such that $Q' \in V'_v$ for all $v \in S$, then $Q = \mathbf{u}^{-1} Q' \in \mathcal{U}(\mathcal{O}_v)$ for all $v \in S' \setminus S$. Therefore Q satisfies $Q \in (\mathcal{X}, \mathcal{M})(B)$ and $Q \in V_v$ for all $v \in S$, as desired.

If K is a number field, we simply let $S' = S$, $Q'_v = Q_v$ and $V'_v = V_v$ for $v \in S \setminus T$.

Choose $\mathbf{m}_1, \ldots, \mathbf{m}_l \in \mathfrak{M}$ such that $\phi(\mathbf{m}_1), \ldots, \phi(\mathbf{m}_l)$ generate N_M as a lattice. If $T = \emptyset$ and $N_M^+ = N$ assume furthermore that they generate N as a monoid. Let $\mathcal{G} \cong \mathbb{G}_{m,\mathbb{Z}}^{n-d}$ be the torus as in Section 3.1 and let $\pi_v \in \mathcal{O}_v$ be a uniformizer for all $v \in S \setminus T$. For a place $v \in S \setminus T$, we are going to construct $c_{\mathbf{m}_1,v}, \ldots, c_{\mathbf{m}_l,v} \in K_v^\times$ such that we have

$$\prod_{s=1}^{l} (c_{\mathbf{m}_s,v}^{m_{s,1}} : \cdots : c_{\mathbf{m}_s,v}^{m_{s,n}}) = Q'_v = (q'_{v,1} : \cdots : q'_{v,n}), \tag{3.2.5}$$

in Cox coordinates, where the multiplication is defined coordinate-wise. This is equivalent to the existence of $(t_1, \ldots, t_n) \in \mathcal{G}(K_v)$ for which

$$\prod_{s=1}^{l} (c_{\mathbf{m}_s,v}^{m_{s,1}}, \ldots, c_{\mathbf{m}_s,v}^{m_{s,n}}) = (t_1, \ldots, t_n) \cdot (q'_{v,1}, \ldots, q'_{v,n}),$$

where the products are again defined by coordinate-wise multiplication.

By the definition of $\mathcal{G}$ this is equivalent to

$$\prod_{i=1}^{n} \left(\left(\prod_{s=1}^{l} c_{\mathbf{m}_s,v}^{m_{s,i}} \right) / q'_{v,i} \right)^{\langle n_{\rho_i}, e_j \rangle} = 1$$

for every $j = 1, \ldots, d$, where $\{e_1, \ldots, e_d\}$ is a basis of N. This, in turn is equivalent to

$$\prod_{s=1}^{l} c_{\mathbf{m}_s,v}^{\langle \phi(\mathbf{m}_s), e_j \rangle} = \prod_{i=1}^{n} {q'_{v,i}}^{\langle n_{\rho_i}, e_j \rangle}$$

for every $j = 1, \ldots, d$. If we write $\gamma_{j,s} = \langle \phi(\mathbf{m}_s), e_j \rangle$ and $a_j = \prod_{i=1}^{n} {q'_{v,i}}^{\langle n_{\rho_i}, e_j \rangle}$ this equation becomes

$$\prod_{s=1}^{l} c_{\mathbf{m}_s,v}^{\gamma_{j,s}} = a_j.$$

The $d \times l$ matrix Γ with entries $\gamma_{j,s}$ induces a group homomorphism $\Gamma_{K_v} : (K_v^\times)^l \to (K_v^\times)^d \cong U(K_v)$ given by

$$(c_1, \ldots, c_l) \mapsto \left(\prod_{s=1}^{l} c_s^{\gamma_{1,s}}, \ldots, \prod_{s=1}^{l} c_s^{\gamma_{d,s}} \right),$$

where the latter isomorphism is given by the choice of basis of N. Since $\phi(\mathbf{m}_1), \ldots, \phi(\mathbf{m}_l)$ span N_M and $|N : N_M| \in \rho(K, C)$, this homomorphism restricts to a surjective group homomorphism $(\mathcal{O}_v^\times)^l \to (\mathcal{O}_v^\times)^d$. Since $K_v^\times \cong \mathbb{Z} \times \mathcal{O}_v^\times$, the image of Γ_{K_v} is exactly the points $\widetilde{Q} \in U(K_v)$ for which $\mathrm{mult}_v(\widetilde{Q}) \in N_M$. Since $\mathrm{mult}_v(q_{v,1} : \cdots : q_{v,n}) \in N_M$ for all $v \in S \setminus T$, for each place $v \in S \setminus T$ we can find $c_{\mathbf{m}_1,v}, \ldots, c_{\mathbf{m}_l,v} \in K_v^\times$ satisfying condition (3.2.5).

Now we distinguish between whether $T \neq \emptyset$ or $T = \emptyset$. If $T \neq \emptyset$, then by Lemma 3.2.16 if K is a global field and by Lemma 3.2.19 if K is another function field, we can find coprime squarefree elements $c_{\mathbf{m}_1}, \ldots c_{\mathbf{m}_l} \in \mathcal{O}(B)$ such that $|c_{\mathbf{m}_i}/c_{\mathbf{m}_i,v} - 1|_v < \epsilon \left(\sum_{s=1}^l m_{s,i}\right)^{-1}$ and $|c_{\mathbf{m}_i}|_v \leq |c_{\mathbf{m}_i,i}|_v$ for every $v \in S \setminus T$.

Therefore if we take $Q' = (q_1', \ldots, q_n') \in (\mathcal{X}, \mathcal{M})(B)$ where

$$q_i' = \prod_{s=1}^l c_{\mathbf{m}_s}^{m_{s,i}} \in \mathcal{O}(B),$$

then $|q_i' - q_{v,i}'|_v = |\prod_{s=1}^l c_{\mathbf{m}_s}^{m_{s,i}} - \prod_{s=1}^l c_{\mathbf{m}_s,v}^{m_{s,i}}|_v < \epsilon |q_{v,i}'|_v$. Here we used the elementary fact that for any tuple $a_1, \ldots, a_r \in K_v$ with $|a_i|_v \leq 1$, we have $|\prod_{i=1}^r a_i - 1|_v \leq \sum_{i=1}^r |a_i - 1|_v$. Therefore (3.2.5) implies that $Q' \in V_v'$ for all $v \in S' \setminus T$, as desired.

Now we assume $T = \emptyset$ and we assume $N = N_M^+$. Without loss of generality we assume that S contains a place v_0, which is an infinite place if K is a number field. Since $\phi(\mathbf{m}_1), \ldots, \phi(\mathbf{m}_l)$ generate N as a monoid, there exist integers $d_1, \ldots, d_l > 0$ such that $\sum_{i=1}^l d_i \phi(\mathbf{m}_i) = 0$. Therefore for the place $v_0 \in S$ we can rescale the constants $c_{\mathbf{m}_i,v_0}$ to $\tilde{c}_{\mathbf{m}_i,v_0} := r^{Cd_i} c_{\mathbf{m}_i,v_0}$ for some integer $C > 0$ and $r \in K_{v_0}$ with $|r|_{v_0} > 1$, without changing the K_{v_0}-rational point defined, since in the torus we have

$$\begin{aligned}\prod_{k=1}^l \left(c_{\mathbf{m}_k,v_0}^{m_{k,1}} : \cdots : c_{\mathbf{m}_k,v_0}^{m_{k,n}}\right) &= \prod_{k=1}^l \left(\left(r^{Cd_k} c_{\mathbf{m}_k,v_0}\right)^{m_{k,1}} : \cdots : \left(r^{Cd_k} c_{\mathbf{m}_k,v_0}\right)^{m_{k,n}}\right) \\ &= \prod_{k=1}^l \left(\tilde{c}_{\mathbf{m}_k,v_0}^{m_{k,1}} : \cdots : \tilde{c}_{\mathbf{m}_k,v_0}^{m_{k,n}}\right)\end{aligned}$$

where the products are defined by the action of the torus on itself. Therefore for every $\epsilon > 0$, by taking C large enough we can apply the stronger form of Lemma 3.2.16 for global fields and Lemma 3.2.19 for other function fields to get a lifting $c_{\mathbf{m}_i} \in \mathcal{O}(B)$ satisfying

$$|c_{\mathbf{m}_i}/c_{\mathbf{m}_i,v} - 1|_v < \epsilon \left(\sum_{s=1}^l m_{s,i}\right)^{-1}$$

for every $v \in S \setminus \{v_0\}$ as before, but with the additional condition that

$$|c_{\mathbf{m}_i} \left(c_{\mathbf{m}_i,v} r^{Cd_i}\right)^{-1} - 1|_v < \epsilon \left(\sum_{s=1}^l m_{s,i}\right)^{-1}.$$

Thus if we define $Q' = (q_1' : \cdots : q_n') \in (\mathcal{X}, \mathcal{M})(B)$ where

$$q_i' = \prod_{k=1}^l c_{\mathbf{m}_k}^{m_{k,i}},$$

as before, then $Q' \in V'_v$ for all $v \in S$ when ϵ is chosen sufficiently small. Therefore (X, M) satisfies M-approximation. □

Proof of necessity of $|N : N_M| \in \rho(K, C)$ *when* $\mathrm{Pic}(C)$ *is finitely generated and* $T \neq \emptyset$. Now we will prove that if $\mathrm{Pic}(C)$ is finitely generated and $|N : N_M| \notin \rho(K, C)$, then (X, M) does not satisfy M-approximation off T for any finite set of places $T \subset \Omega_K$. Since $\mathrm{Pic}(C)$ is finitely generated, we can find a finite set of places $S \subset \Omega_K$ containing the infinite places such that $B = \Omega_K \setminus S$ satisfies $\mathrm{Pic}(B) = 1$. We will show that this implies that for every finite set of places $T \subset S$, the toric integral model $(\mathcal{X}, \mathcal{M})$ does not satisfy integral $\mathcal{M}$-approximation off T. By Proposition 2.2.12, this in turn implies that (X, M) does not satisfy M-approximation off T. Since every finite set of places T is contained in a subset S with $\mathrm{Pic}(B) = 1$, (X, M) does not satisfy M-approximation off T for every finite set of places T.

Let $\mathbf{m}_1, \dots, \mathbf{m}_l$ generate N_M as before in the proof of the sufficiency of the condition. Additionally, we first assume that $|N : N_M|$ is not a power of the characteristic of K. By Proposition 3.2.4, the image of the set $(\mathcal{X}, \mathcal{M})(B) \subset X(K)$ is contained in the image of the map

$$\mathcal{U}(B) \times (\mathcal{O}(B) \setminus \{0\})^l \to X(K)$$

given by

$$((u_1 : \dots : u_n), (a_1, \dots, a_l)) \mapsto \left(u_1 \prod_{i=1}^{l} a_i^{m_{i,1}} : \dots : u_n \prod_{i=1}^{l} a_i^{m_{i,n}} \right),$$

where $\mathcal{U} \cong \mathbb{G}^d_{m,\mathbb{Z}}$ is the open torus in $\mathcal{X}$. In particular, if $(\mathcal{X}, \mathcal{M})$ were to satisfy integral $\mathcal{M}$-approximation off T, then the induced map

$$g_{S'} : \mathcal{U}(B) \times \prod_{v \in S'} (k_v^\times)^l \to \prod_{v \in S'} \mathcal{U}(k_v)$$

would be surjective for any finite set of places $S' \subset \Omega_K \setminus S$.

As in the proof of the sufficiency of $N = N_M$ for M-approximation off T, let $\{e_1, \dots, e_d\}$ be a basis of N and Γ be the $d \times l$ matrix with coefficients $\gamma_{j,s} = \langle \phi(\mathbf{m}_s), e_j \rangle$. The isomorphism $N \cong \mathbb{Z}^d$ induced by the choice of this basis induces an isomorphism $\mathcal{U}(k_v) \cong (k_v^\times)^d$ for all $v \in \Omega_K$. Under this isomorphism, the homomorphism $(k_v^\times)^l \to \mathcal{U}(k_v)$

$$(a_1, \dots, a_l) \mapsto \left(\prod_{i=1}^{l} a_i^{m_{i,1}} : \dots : \prod_{i=1}^{l} a_i^{m_{i,n}} \right)$$

is given by the homomorphism $\Gamma_{k_v} : (k_v^\times)^l \to (k_v^\times)^d$ induced by the matrix Γ. Since $|N : N_M| \notin \rho(K, C)$, Γ_{k_v} is not surjective for infinitely many choices of $v \in \Omega_K$. If K is a global field, then $\mathcal{U}(B)$ is finitely generated, say by t elements. Then $g_{S'}$ cannot be surjective for any finite set of places S' containing strictly more than t places $v \in \Omega_K \setminus S$ for which Γ_{k_v} is not surjective.

Now assume that K is a function field of a curve C over a field k. If $g_{S'}$ is surjective, then the induced homomorphism

$$\{f \in \mathcal{U}(B) \mid f(v') = (1 : \dots : 1)\} \times \prod_{v \in S' \setminus \{v'\}} (k_v^\times)^l \to \prod_{v \in S' \setminus \{v'\}} \mathcal{U}(k_v)$$

is also surjective, where $f(v')$ is the image of f in $\mathcal{U}(k_{v'})$. We now give an analogous argument as for global fields. The group $\{f \in \mathcal{U}(B) \mid f(v') = (1 : \cdots : 1)\}$ injects into $\mathcal{U}(B)/\mathcal{U}(k)$, since if $f(v') = (1 : \cdots : 1)$ and $g \in \mathcal{U}(k) \setminus \{(1 : \cdots : 1)\}$, then $(f \cdot g)(v') \neq (1 : \cdots : 1)$. Therefore, since $\mathcal{U}(B)/\mathcal{U}(k) \cong (\mathcal{O}(B)^\times/\mathcal{O}(k)^\times)^d$ is finitely generated, the group $\{f \in \mathcal{U}(B) \mid f(v') = (1 : \cdots : 1)\}$ is finitely generated as well. Suppose that it is generated by t elements, then $g_{S'}$ cannot be surjective as soon as it contains strictly more than $t+1$ places v for which Γ_v is not surjective.

Finally, if $|N : N_M|$ is a power of the characteristic of K, the argument given above might not work, since in this case Γ_{k_v} could be surjective for all $v \in \Omega_K$. However there are still infinitely many places $v \in \Omega_K$ such that the map $(\mathcal{O}_v^\times)^l \to (\mathcal{O}_v^\times)^d$ induced by Γ is not surjective. Therefore, the argument is easily amended by replacing the role of k_v by $\mathcal{O}_v/\pi_v^{n_v}$ for these places v, where π_v is a uniformizer and n_v is the least positive integer such that the homomorphism $((\mathcal{O}_v/\pi_v^{n_v})^\times)^l \to ((\mathcal{O}_v/\pi_v^{n_v})^\times)^d$ is not surjective.

□

Proof of necessity of $N = N_M^+$ when $T = \emptyset$. Now we will show that if (X, M) satisfies M-approximation, then $N_M^+ = N$. We argue by contradiction and assume $N_M^+ \neq N$. Then we have $N_M \neq N$ or $N_{M,\mathbb{R}}^+ \neq N \otimes_{\mathbb{Z}} \mathbb{R}$, where $N_{M,\mathbb{R}}^+$ is the convex cone generated by N_M^+. This follows from the fact that $N_{M,\mathbb{R}}^+ = N \otimes_{\mathbb{Z}} \mathbb{R}$ implies that N_M^+ contains a lattice of finite index in N, so combined with $N_M = N$ this gives $N_M^+ = N$.

First we assume $N \neq N_M$. If K is a global field, then we have seen that this assumption implies that (X, M) does not satisfy M-approximation off T for T nonempty, since then $\rho(K, C) = 1$. Thus we assume that K is a function field of a curve, and we consider the map $U(K) \to N$ given by $P \mapsto \sum_{v \in \Omega_K} \phi_v(P)$, where ϕ_v is defined as in (3.2.1). By Proposition 3.2.4 this map is identically zero, as $\deg \operatorname{div}(f) = 0$ in $\operatorname{Pic}(C)$ for any $f \in K^\times$. Furthermore, if $P \in (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$ for some place $v \in \Omega_K^{<\infty}$, then $\phi_v(P) \in N_M$ by definition of N_M. In particular, if $\phi_{v'}(Q_{v'}) = 0$ for some place $v' \in S$ and $a \in N \setminus N_M$, and $\phi_v(Q_v) = 0$ for all $v \in S \setminus \{v'\}$, then these points cannot be all simultaneously well approximated by some $Q \in (\mathcal{X}, \mathcal{M})(B)$. This is because the equality $\phi_v(Q) = \phi_v(Q_v)$ for all $v \in S$ is incompatible with $\sum_{v \in \Omega_K} \phi_v(Q) = 0$. Thus (X, M) does not satisfy M-approximation.

Now we assume $N_{M,\mathbb{R}}^+ \neq N \otimes_{\mathbb{Z}} \mathbb{R}$. The cone $N_{M,\mathbb{R}}^+$ is contained in some half space H of $N \otimes_{\mathbb{Z}} \mathbb{R}$, which is given as

$$H = \left\{ \sum_{i=1}^d x_i e_i \,\middle|\, \sum_{i=1}^d a_i x_i \geq 0 \right\} \subset N \otimes_{\mathbb{Z}} \mathbb{R},$$

where the $e_1, \ldots, e_d$ form a basis of N and $a_1, \ldots, a_d \in \mathbb{R}$, not all zero.

Let S be a nonempty set of places containing Ω_K^∞ and let $Q \in (\mathcal{X}, \mathcal{M})(B)$. Under the isomorphism $\mathcal{U}(K) \cong (K^\times)^d$ induced by the choice of basis of N given above, we can write $Q = (x_1, \ldots, x_d) \in (K^\times)^d$. Since $Q \in (\mathcal{X}, \mathcal{M})(B)$, we have by Proposition 3.2.4 that $\sum_{i=1}^d a_i v(x_i) \geq 0$ for every $i \in \{1, \ldots, d\}$ and every place $v \in \Omega_K \setminus S$. Furthermore, the product formula gives $\prod_{v \in \Omega_K} |x_i|_v = 1$ and thus we see $\prod_{v \in S} \prod_{i=1}^d |x_i|_v^{a_i} \geq 1$. However for $v \in S$, we can consider points $Q_v \in X(K_v)$ which are sent to $(x_{v,1}, \ldots, x_{v,d}) \in (K_v^\times)^d$ under the isomorphism induced by N, such that $\prod_{v\in S}\prod_{i=1}^{d}|x_{v,i}|_v^{a_i} < \frac{1}{2}$. Such a tuple $(Q_v)_{v\in S}$ cannot lie in the closure of the map $(\mathcal{X},\mathcal{M})(B) \to \prod_{v\in S} U(K_v)$, and hence (X, M) does not satisfy M-approximation. □

3.2.4 Integral $\mathcal{M}$-approximation on toric varieties

For completeness, we also characterize when integral $\mathcal{M}$-approximation holds on a toric variety.

Proposition 3.2.20. *Let X be a complete normal split toric variety and let $T \subset \Omega_K$ be a finite set of places. If (X, M) satisfies M-approximation off T, then the toric integral model $(\mathcal{X},\mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T if and only if $\mathfrak{M}_{red}$ is contained in the closure of $\mathfrak{M}_{\text{fin}}$.*

Proof. By Proposition 2.2.12, $(\mathcal{X},\mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T if and only if $(\mathcal{X},\mathcal{M}_{\text{fin}})(\mathcal{O}_v)$ is dense in $(\mathcal{X},\mathcal{M})(\mathcal{O}_v)$ for all $v \in \Omega_K \setminus T$ and by Corollary 3.2.1 we can assume X is smooth. By Proposition 2.1.2 mult_v is continuous, so a point $P \in (\mathcal{X},\mathcal{M})(\mathcal{O}_v)$ can only lie in the closure of $(\mathcal{X},\mathcal{M}_{\text{fin}})(\mathcal{O}_v)$ if $\text{mult}_v(P)$ lies in the closure of $\mathfrak{M}_{\text{fin}}$. Since the image of $(\mathcal{X},\mathcal{M})(\mathcal{O}_v)$ under mult_v is $\mathfrak{M}_{\text{red}}$, this shows that integral $\mathcal{M}$-approximation off T can only hold if $\mathfrak{M}_{\text{red}}$ is contained in the closure of $\mathfrak{M}_{\text{fin}}$. Conversely, if $P = (a_1\pi^{m_1} : \cdots : a_n\pi^{m_n})$ with $\pi \in \mathcal{O}_v$ a uniformizer, $a_1, \dots, a_n \in \mathcal{O}_v^{\times}$, and $(m_1, \dots, m_n) \in \overline{\mathfrak{M}_{\text{fin}}}$, we can choose a sequence $((m_{1,j}, \dots, m_{n,j}))_{j\in\mathbb{N}}$ in $\mathfrak{M}_{\text{fin}}$ converging to $(m_1, \dots, m_n)$. By setting $P_j = (a_1\pi^{m_{1,j}} : \cdots : a_n\pi^{m_{n,j}}) \in (\mathcal{X},\mathcal{M}_{\text{fin}})(\mathcal{O}_v)$ we obtain a sequence converging to P, finishing the proof. □

3.3 The $\mathcal{M}$-Hilbert property and split toric varieties

While the previous section considered the situation where $\mathcal{M}$-points are plentiful, in this section we will consider when the set of such points is thin. We will also investigate the different degrees of thinness that these sets have. For example, by Dirichlet's unit theorem [Nar04, Theorem 3.12], $\mathbb{G}_m(\mathcal{O}_K)$ is finitely generated if K is a number field, while the set of squares in $\mathbb{G}_m(K)$ is not finitely generated as a group. Therefore, the former can be thought of as 'thinner' than the latter. We introduce several variants of thinness, which allows us to make this idea precise.

Definition 3.3.1. Let X be an integral variety over K, let $A \subset X(K)$ and let $d > 1$ be an integer. We say that A is of *type II(d)* if there is an integral variety Y with $\dim Y = \dim X$ and a generically finite morphism $f\colon Y \to X$ of degree $\geq d$ such that $A \subset f(Y(K))$. We say that A is *d-thin* if it is a finite union of sets of type I and II(d), where type I is defined as in Definition 2.2.16. We say that A is *strictly d-thin* if the morphisms f have degree exactly d.

We also introduce a notion of thinness which is preserved under taking inverse images of dominant morphisms.

Definition 3.3.2. Let K be a field, X an integral variety and let $A \subset X(K)$ be a subset. Then we say that A is *stably thin* if for every dominant morphism $f\colon Y \to X$ of integral varieties over K, $f^{-1}A \subset Y(K)$ is thin.

This property is preserved under many operations. For example, in the above situation, $f^{-1}A$ is also stably thin. A stably thin set can be viewed as a sort of "∞-thin" set, as the next proposition shows.

Proposition 3.3.3. *Let K be a field, X an integral variety and let $A \subset X(K)$ be a subset. Then A is stably thin if and only if it is d-thin for every $d > 1$.*

Proof. We first prove by induction that if A is stably thin, then it is d-thin for every $d > 1$. Assume that A is stably thin and that A is d-thin for some $d > 1$. Then there is an integer $n > 1$ and for each $i \in \{1, \dots, n\}$ a morphism $f_i\colon Y_i \to X$ of integral K-varieties of degree at least d such that $A \setminus A' \subset \bigcup_{i=1}^{n} f_i(Y_i(K))$, for some subset $A' \subset A$ which is not Zariski dense in X. Since A is stably thin, $B_i\colon = f_i^{-1}(A)$ is thin for each $i \in \{1, \dots, n\}$, so we see that A is $2d$-thin. Since any stably thin set is 2-thin, this implies by induction that any stably thin set is d-thin for every integer $d > 1$.

Conversely, assume A is d-thin for every $d > 1$ and let $g\colon Z \to X$ be a dominant morphism of integral varieties over K. Then the algebraic closure of the function field $K(X)$ in $K(Z)$ is a finite extension of degree l. For a generically finite morphism $f\colon Y \to X$ of degree $d > l$, and for every irreducible component Y' of $(Y \times_X Z)_{\mathrm{red}}$, consider the induced morphism $Y' \to Z$. This is a generically finite morphism, since the restriction $f^{-1}U \to U$ is finite for some dense open $U \subset X$ so $(f^{-1}U \times_X Z)_{\mathrm{red}} \to Z \times_X U$ is finite. The degree of $Y' \to Z$ is at least 2, since $K(Y')$ contains a finite extension of $K(X)$ of degree d, which cannot be contained in $K(Z)$.

Since A is d-thin, there exists an integer n, a subset $A' \subset A$ which is not Zariski dense and for $i \in \{1, \dots, n\}$ a dominant generically finite morphism $f_i\colon Y_i \to X$ of integral K-varieties such that $A \setminus A' \subset \bigcup_{i=1}^{n} f_i(Y_i(K))$ and each f_i has degree at least $d > l$. Denote the irreducible components of $(Y_i \times_X Z)_{\mathrm{red}}$ by Y'_{ij} and the induced morphisms to Z by $f_{ij}\colon Y'_{ij} \to Z$. The set $g^{-1}A'$ is not Zariski dense in Z, and by construction $g^{-1}(A \setminus A') \subset \bigcup_{i,j} f_{ij}(Y'_{ij}(K))$. Thus we see that $g^{-1}A$ is thin and therefore A is stably thin. □

Rational points on abelian varieties and integral points on tori give examples of stably thin sets, as the next example shows.

Example 3.3.4. Let A be a finitely generated subgroup of $G(K)$ for an semiabelian variety G of positive dimension over a field K. For example, $A = G(K)$ when K is a global field and G is an abelian variety. Then for any integer $d > 1$, the group A/dA is finite. Let $a_1, \dots, a_n \in A$ be a set of representatives for the classes in A/dA. If $a \in dA + a_i$ for some $i \in \{1, \dots, n\}$, then it is the image of a K-point under the morphism $G \to G$ given by multiplication by d followed by translating by a_i. This morphism has degree divisible by d, so A is d-thin. As d was arbitrary, this implies that A is stably thin.

This notion gives a way to formalize the intuition that for a number field K there are more integer squares in $\mathbb{G}_m(K)$ than integral units $\mathbb{G}_m(\mathcal{O}_K) \subset \mathbb{G}_m(K)$. The former is thin, but probably not 3-thin, while the latter is stably thin.

Theorem 3.3.5. *Let (K, C) be a PF field such that $\operatorname{Pic}(C)$ is finitely generated and let $d > 1$ be an integer. If K is a function field assume that $(k^\times)/(k^\times)^d$ is finite. Let (X, M) be a toric pair where X is a normal complete split toric variety over K and let $(\mathcal{X}, \mathcal{M})$ be any integral model over B. Let N_M be the lattice as in Definition 3.2.6. Then*

1. *if N_M has finite index in N and d divides $|N : N_M|$, then $(\mathcal{X}, \mathcal{M})(B) \subset X(K)$ is strictly d-thin.*
2. *if N_M does not have finite index in N, then $(\mathcal{X}, \mathcal{M})(B)$ is stably thin.*
3. *in the function field case, if $(\mathcal{X}, \mathcal{M})$ is the toric integral model and $N_M^+ \neq N_M$, then $(\mathcal{X}, \mathcal{M})(C) \subset X(K)$ is stably thin.*

Furthermore, if $(\mathcal{X}, \mathcal{M})$ is the toric integral model and $\mathbb{G}_m(B)$ is finite, then $(\mathcal{X}, \mathcal{M})(B)$ is not Zariski dense in X if and only if $(\mathcal{X}, \mathcal{M})(B)$ is stably thin.

Remark 3.3.6. If (K, C) is a function field of a curve and $d \in \rho(K, C)$, then $(k^\times)/(k^\times)^d$ is trivial or has order 2. If k is perfect of characteristic p, then $(k^\times)^p = k^\times$.

Remark 3.3.7. If the group $\mathbb{G}_m(B)$ is infinite, then the points on the toric integral model $(\mathcal{X}, \mathcal{M})(B)$ are always Zariski dense, so Theorem 3.3.5 completely characterizes when the $\mathcal{M}$-points on a toric integral model are Zariski dense. Furthermore, the group $\mathbb{G}_m(B)$ is finite if and only if $K = \mathbb{Q}$ or an imaginary quadratic number field and $B = \operatorname{Spec} \mathcal{O}_K$, or $K = k(C)$ for k a finite field and $C \setminus B$ contains at most one point.

Proof. Note that in the first two statements B can be chosen as small as we want, since $B' \subset B$ implies $(\mathcal{X}, \mathcal{M})(B') \supset (\mathcal{X}, \mathcal{M})(B)$. Thus it follows from Proposition 2.1.25 that we can assume without loss of generality that $(\mathcal{X}, \mathcal{M})$ is the toric integral model, and for the first two statements we can assume that $\operatorname{Pic}(B)$ is trivial. We can also assume that $M = M_{\text{fin}}$ without loss of generality. Since for a toric resolution of singularities $f \colon Y \to X$ with $f^{-1}D_i$ Cartier for all $i = 1, \dots, n$, the set $(\mathcal{X}, \mathcal{M})(B)$ is thin if and only if $(\mathcal{Y}, f^*\mathcal{M})(B)$ is thin, we can assume that X is smooth.

Assume that $N_M \neq N$, $|N : N_M|$ is finite, $d > 1$ divides $|N : N_M|$ and K is a global field or $(k^\times)/(k^\times)^d$ is finite. Then $\mathcal{U}(B)/\mathcal{U}(B)^d$ is finite, where $\mathcal{U} = \mathbb{G}_{m,B}^{\dim X}$, since $\mathcal{O}(B)^\times$ is finitely generated when K is a global field and $\mathcal{O}(B)^\times/k^\times$ is finitely generated when K is a function field. There exists a lattice N' such that $N_M \subset N' \subset N$ such that $|N : N'| = d$ and so, using the surjectivity of ϕ proven in Proposition 3.2.4, we can choose $M \subset M'$ such that $|N : N_{M'}| = d$. The inclusion $M \subset M'$ implies $(\mathcal{X}, \mathcal{M})(B) \subset (\mathcal{X}, \mathcal{M}')(B)$, and thus it suffices to consider the case $d = |N : N_M|$. By intersecting the fan of X with N_M, we get a new complete normal split toric variety X_M and a degree d morphism $X_M \to X$. By a resolution of singularities $X' \to X_M$, we find a degree d morphism $f \colon X' \to X$ of smooth complete split toric varieties. By Proposition 3.2.4 it follows that for every place $v \in B$ and point $P \in (\mathcal{X}, \mathcal{M})(\mathcal{O}_v)$, there exists $P' \in (\mathcal{X}', f^*\mathcal{M})(\mathcal{O}_v) \subset X'(K_v)$ such that $\phi_v(P) = \phi_v(f(P'))$. Since we assumed $\operatorname{Pic}(B)$ is trivial, this means that for all $P \in (\mathcal{X}, \mathcal{M})(B)$ there exists $P' \in X'(K)$ such that $\phi_v(P) = \phi_v(f(P'))$ for all $v \in B$. Thus we see by Proposition 3.2.4 that the image of f contains an element from every $\mathcal{U}(B)$-orbit in $(\mathcal{X}, \mathcal{M})(B)$. The image of $\mathcal{U}'(B)$ in $\mathcal{U}(B)$, where $\mathcal{U}'$ is the torus in

$\mathcal{X}'$, contains $\mathcal{U}(B)^d \subset \mathcal{U}(B)$ and by the assumption on d we know that $\mathcal{U}(B)/\mathcal{U}(B)^d$ is finite. Therefore $H = \mathcal{U}(B)/f(\mathcal{U}'(B))$ is finite, and by choosing representatives $u_1, \ldots, u_r \in \mathcal{U}(B)$ for H, we find degree d morphisms

$$f_i \colon X' \to X$$

defined by $P' \mapsto u_i f(P')$ for $P' \in X'(K)$, so that every point in $(\mathcal{X}, \mathcal{M})(B)$ lies in the image of one of the f_i. This proves the first statement.

Now assume that N_M does not have finite index in N. Then there exists an embedding $N_M \to \mathbb{Z}^{\dim X - 1} \times \{0\} \subset \mathbb{Z}^{\dim X} \cong N$. By Proposition 3.2.4, this implies that there is an embedding $(\mathcal{X}, \mathcal{M})(B) \subset \mathbb{G}_m^{\dim X - 1}(K) \times \mathbb{G}_m(B) \subset \mathbb{G}_m^{\dim X}(K)$. Since $\mathbb{G}_m(B)$ is finitely generated if K is a global field, and otherwise $\mathbb{G}_m(B)/k^\times$ is finitely generated, $\mathbb{G}_m(B)/\mathbb{G}_m(B)^{d^l}$ is finite for every integer $l > 0$. Thus $\mathbb{G}_m(B)$ is d^l-thin for every $l > 0$ and therefore by Proposition 3.3.3 implies that it is a stably thin subset of $\mathbb{G}_m(K)$. Therefore $(\mathcal{X}, \mathcal{M})(B)$ is also a stably thin subset of $X(K)$. If furthermore $\mathbb{G}_m(B)$ is finite and thus not Zariski dense in $\mathbb{G}_m$, then $(\mathcal{X}, \mathcal{M})(B)$ is not Zariski dense in X.

Now we prove the third statement, so we assume that $B = C$ and $N_M^+ \neq N_M$. Let N' be the largest lattice contained in N_M^+. N' does not have finite index in N since otherwise the cone generated by N_M^+ would be $N_\mathbb{R}$, which implies $N_M^+ = N_M$. Since any $P \in U(K)$ satisfies $\sum_{v \in \Omega_K} \phi_v(P) = 0$ by the product formula, any $P \in (\mathcal{X}, \mathcal{M})(C)$ satisfies $\phi_v(P) \in N'$ for all $v \in \Omega_K$. Let (X, M') be the largest toric pair contained in (X, M) such that $\phi_v(P) \in N'$ for all $P \in (\mathcal{X}, \mathcal{M}')(\mathcal{O}_v)$. Then $(\mathcal{X}, \mathcal{M}')(C)$ contains $(\mathcal{X}, \mathcal{M})(C)$ and it is a stably thin subset of $X(K)$ since $N' = N_{M'}$ does not have finite index in N. If $\mathbb{G}_m(C)$ is finite, then $(\mathcal{X}, \mathcal{M}')(C)$ is not Zariski dense in X, and thus $(\mathcal{X}, \mathcal{M})(C)$ is not Zariski dense in X either. □

Remark 3.3.8. The assumption on the finiteness of $(k^\times)/(k^\times)^d$ in Theorem 3.3.5 is satisfied by many fields, such as

1. d-closed fields, such as separably closed fields if $\mathrm{char}(k) \nmid d$,
2. perfect fields if d is a power of $\mathrm{char}(k)$,
3. finite fields,
4. real closed fields,
5. local fields if $\mathrm{char}(k) \nmid d$ [CF67, Chapter I, Section 1, Proposition 5],
6. Euclidean fields if d is a power of 2.

Remark 3.3.9. The generically finite morphisms used in the proof of Theorem 3.3.5 are ramified since any complete toric variety is geometrically simply connected [SGA1, Exposé XI, Corollaire 1.2]. Therefore the thin sets in the theorem are strongly thin as defined in [BFP23].

We now prove Corollary 1.1.9 by specializing Theorem 3.3.5 to global fields.

Proof of Corollary 1.1.9. First assume $T \neq \emptyset$. By Theorem 1.1.3, the toric pair (X, M) satisfies M-approximation off T if and only if $N = N_M$. By Theorem 1.1.1 this implies the $\mathfrak{M}$-Hilbert property over B for any integral model $(\mathcal{X}, \mathcal{M})$ of (X, M) satisfying $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$. On the other hand Theorem 3.3.5 implies that $(\mathcal{X}, \mathcal{M})(B)$ is thin if $N \neq N_M$, so $(\mathcal{X}, \mathcal{M})(B)$ is thin if and only if $N \neq N_M$.

Now we assume $T = \emptyset$. Then Theorem 1.1.3 implies that the toric pair (X, M) satisfies M-approximation if and only if $N_M^+ = N$. If $(\mathcal{X}, \mathcal{M})$ is the toric integral model of (X, M) and $N \neq N_M^+$, then Theorem 3.3.5 implies that $(\mathcal{X}, \mathcal{M})(B)$ is stably thin. $\square$

In Theorem 3.3.5, the condition on the finiteness of $(k^\times)/(k^\times)^d$ is necessary for the result to be true, as the next proposition shows that the $\mathcal{M}$-Hilbert property is always satisfied on a split toric variety for a function field $K = k(C)$, where k is a Hilbertian field of characteristic zero. See [FJ05, Chapter 12] for background on Hilbertian fields.

Proposition 3.3.10. *Let (K, C) be a PF field, where C is a curve over an Hilbertian field k of characteristic zero. Let X be a proper integral variety over k with $X(k)$ not thin and let $\mathcal{X} = X \times_k C$. Let (X, M) be a pair over (K, C) such that $D_\alpha \neq X$ for all $\alpha \in \mathcal{A}$. Then the set $(\mathcal{X}, \mathcal{M}^c)(C) \subset X(K)$ is not thin.*

Proof. As the closed subschemes D_α are all proper closed subsets of X, $U = X \setminus_{\alpha \in \mathcal{A}} D_\alpha$ is a dense open. The constant sections of $U \times C \to C$ correspond to the k-rational points on U. These are $\mathcal{M}$-points, as they avoid the closed subschemes $\mathcal{D}_\alpha = D_\alpha \times_k C$ entirely so $U(k) \subset (\mathcal{X}, \mathcal{M}^c)(C)$. By [BFP23, Theorem 1.1], $U(k)$ is not a thin subset of $X(K)$ since K/k is a finitely generated extension. Thus $(\mathcal{X}, \mathcal{M}^c)(C)$ is not thin. $\square$

Using Theorem 1.1.3 and Theorem 3.3.5, we can produce examples over some PF fields where integral $\mathcal{M}$-approximation does not imply the $\mathcal{M}$-Hilbert property, in contrast to the situation over global fields.

Corollary 3.3.11. *Let (K, C) be a PF field with $\rho(K, C) \neq 1$, let $T \subset \Omega_K$ be a nonempty finite set of places, and set $B = C \setminus (T \cap C)$. Let (X, M) be a toric pair with $M = M_{\text{fin}}$ and $|N : N_M| \in \rho(K, C)$, $N_M \neq N$. Then the toric integral model $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T, but $(\mathcal{X}, \mathcal{M})(B)$ is thin.*

Proof. Combine Theorem 1.1.3 and Theorem 3.3.5 together with the observation that the ground field of C satisfies $k^\times = (k^\times)^{|N:N_M|}$ by Lemma 3.2.12 since $|N : N_M| \in \rho(K, C)$. $\square$

Example 3.3.12. For any smooth split toric variety U over K such that $\mathrm{Pic}(U)$ contains torsion, the B-integral points on the toric integral model $\mathcal{U}$ are thin for any nonempty open $B \subset C$. This follows from combining Theorem 3.3.5 with Proposition 3.1.7. However, as we will see in Corollary 3.4.2, U still satisfies strong approximation off any single place if the orders of torsion in $\mathrm{Pic}(U)$ are contained in $\rho(K, C)$ and $\mathcal{O}(X_{\overline{K}})^\times = \overline{K}^\times$.

Corollary 3.3.11 leaves several natural questions on potential extensions of Theorem 1.1.1.

Question 3.3.13. Let (K, C) be a PF field with $\rho(K, C) = \{1\}$. Let (X, M) be a pair over (K, C) with integral model $(\mathcal{X}, \mathcal{M})$ over $B \subset C$, such that X is a geometrically integral variety and $D_\alpha \neq X$ for any $\alpha \in \mathcal{A}$. Suppose that $(\mathcal{X}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off a finite set of places $T \subset \Omega_K$ and $(\mathcal{X}, \mathcal{M})(B) \neq \emptyset$. Does $(\mathcal{X}, \mathcal{M})$ satisfy the $\mathcal{M}$-Hilbert property over B?

In order to obtain pairs satisfying integral $\mathcal{M}$-approximation but failing the $\mathcal{M}$-Hilbert property in Corollary 3.3.11, we needed to take $T \neq \emptyset$, so it also makes sense to ask the following variant of the previous question.

Question 3.3.14. Let (K, C) be a PF field with $\rho(K, C) \neq \{1\}$ and let $T = \emptyset$. With the other assumptions as in Question 3.3.13, does $(\mathcal{X}, \mathcal{M})$ satisfy the $\mathcal{M}$-Hilbert property over B?

In the setting of split toric varieties, it seems likely that the results for the $\mathcal{M}$-Hilbert property should extend, so we pose the following conjecture.

Conjecture 3.3.15. *Let (K, C) be a PF field and let (X, M) be a toric pair. If $N_M = N$, then the toric integral model $(\mathcal{X}, \mathcal{M})$ satisfies the $\mathcal{M}$-Hilbert property over any open $B \subsetneq C$. If furthermore $N_M^+ = N$, then $(\mathcal{X}, \mathcal{M})$ satisfies the $\mathcal{M}$-Hilbert property over C.*

3.4 Strong approximation and M-approximation for Campana points

In this section we will consider special cases of Theorem 1.1.3 and its implications for integral points, Campana points and Darmon points on split toric varieties.

If a toric pair (X, M) encodes the integrality condition of an open $V \subset X$, then the lattice N_M is related to the fundamental group of V.

Proposition 3.4.1. *Let K be a PF field of characteristic 0, let X be a complete normal split toric variety over K and let $V \subset X$ be an open toric subvariety.*

If (X, M) is the toric pair corresponding to $V \subset X$ as in the first example 1 of Section 2.1.4, then there exists an isomorphism of profinite groups

$$\pi_1(V_{\overline{K}}) \cong \widehat{N/N_M},$$

where $\pi_1(V_{\overline{K}})$ is the étale fundamental group of $V_{\overline{K}}$ and $\widehat{N/N_M}$ is the profinite completion of N/N_M.

Furthermore, the only regular functions on $V_{\overline{K}}$ are constant if and only if the cone generated by N_M^+ is $N_{\mathbb{R}}$.

Proof. By [Stacks, Tag 0A49] we have a natural isomorphism $\pi_1(V_{\overline{K}}) \cong \pi_1(V_{\mathbb{C}})$ of étale fundamental groups, where $V_{\mathbb{C}}$ is the toric variety over $\mathbb{C}$ with the same fan as U. Now the isomorphism follows directly from [CLS11, Theorem 12.1.10].

Note that the cone generated by N_M^+ is exactly the support $|\Sigma_V|$ of the fan Σ_V defining V, as defined in [CLS11, Definition 3.1.2]. By [CLS11, Exercise 4.3.4], which generalizes to arbitrary fields, $|\Sigma| = N_{\mathbb{R}}$ is equivalent to $\mathcal{O}(V_{\overline{K}}) = \overline{K}$. □

Proof of Corollary 1.1.6. Combine Proposition 3.4.1 with Theorem 1.1.3. □

The description of the fundamental group given in Proposition 3.4.1 does not hold if K has positive characteristic, since then even the affine line $\mathbb{A}^1_{\overline{K}}$ has an infinite fundamental group (see e.g. [Kum14, Theorem 1.1]). Nevertheless, a similar weaker result is still true if one considers only covers of degree coprime to $\mathrm{char}(K)$, see Remark 3.5.9. The following corollary gives another characterisation of strong approximation, which is also valid in positive characteristic.

Corollary 3.4.2. *Let (K, C) be a PF field, let V be a smooth split toric variety, and let $T \subset \Omega_K$ be a nonempty finite set of places.*

1. *The variety V satisfies strong approximation off T if $\mathcal{O}(X_{\overline{K}})^\times = \overline{K}^\times$ and $|\operatorname{Pic}(V)_{\mathrm{tors}}| \in \rho(K, C)$, where $\operatorname{Pic}(V)_{\mathrm{tors}}$ is the torsion subgroup of $\operatorname{Pic}(V)$. The converse also holds if $\operatorname{Pic}(C)$ is finitely generated.*

2. *The variety V satisfies strong approximation if and only if $\mathcal{O}(V) = K$ and $\operatorname{Pic}(V)$ is torsion-free.*

Proof. Choose a smooth toric compactification $V \subset X$. The first claim is a direct consequence of Proposition 3.1.7 and Theorem 1.1.3, while the second one follows from combining these results with [CLS11, Exercise 4.3.4] (which holds over general fields). The last claim follows from [CLS11, Proposition 4.2.5] (where we note that the result is independent of the field.) □

Remark 3.4.3. By [CLS11, Proposition 4.2.5], which generalizes to arbitrary fields, any toric variety whose fan contains a cone of maximal dimension has a torsion-free Picard group. By using a resolution of singularities, Corollary 3.4.2 implies that a normal affine toric variety V satisfies strong approximation off a nonempty set of places T if and only if it does not have torus factors.

Over number fields, we have yet another characterisation of strong approximation for toric varieties.

Corollary 3.4.4. *Let K be a number field, let V be a smooth split toric variety and let $T \subset \Omega_K$ be a nonempty finite set of places. Then the following are equivalent:*

1. *The variety V satisfies strong approximation off T.*

2. $\operatorname{Br}(V)/\operatorname{Br}_0(V) = 0$.

3. $\operatorname{Br}_1(V)/\operatorname{Br}_0(V) = 0$.

Here, $\operatorname{Br}(V)$ is the Brauer group of V, $\operatorname{Br}_1(V) = \ker(\operatorname{Br}(V) \to \operatorname{Br}(V_{\overline{K}}))$ is the algebraic Brauer group, and $\operatorname{Br}_0(V) = \operatorname{im}(\operatorname{Br}(K) \to \operatorname{Br}(V))$ consists of the constant elements in $\operatorname{Br}(V)$. If V satisfies any of the above conditions (1)-(3), then V satisfies strong approximation if and only if $\mathcal{O}(V) = K$.

Proof. We choose a smooth toric compactification $V \subset X$. If (X, M) is the pair corresponding to integral points on V, then by Theorem 1.1.3, V satisfies strong approximation if and only if $N = N_M$. By [DF93, Corollary 1.3], $N = N_M$ implies that the transcendental Brauer group $\operatorname{Br}(V_{\overline{K}}) \cong \operatorname{Br}(V)/\operatorname{Br}_1(V)$ is trivial.

If V has a torus factor, then $V = V' \times \mathbb{G}_m$ for some split toric variety V'. So since $\operatorname{Br}_1(\mathbb{G}_m^1)/\operatorname{Br}_0(\mathbb{G}_m^1)$ is nontrivial, it follows that $\operatorname{Br}_1(V)/\operatorname{Br}_0(V) \ncong 0$. If V does not have torus factors, then [CS21, Proposition 5.4.2, Remark 5.4.3(3)] implies that $\operatorname{Br}_1(V)/\operatorname{Br}_0(V) \cong H^1(K, \operatorname{Pic} V_{\overline{K}})$. The Galois cohomology group $H^1(K, \operatorname{Pic} V_{\overline{K}})$ is trivial if and only if $\operatorname{Pic} V_{\overline{K}} \cong \operatorname{Pic} V$ is torsion-free, since $\operatorname{Gal}(\overline{K}/K)$ acts trivially on $\operatorname{Pic} V$. Now the equivalence of the statements follows from Corollary 3.4.2. □

Using the above criteria for strong approximation, we can characterize when M-approximation is satisfied for Campana points. Now we will use Theorem 1.1.3 to prove Corollary 1.1.8, characterising M-approximation for Campana points.

Proof of Corollary 1.1.8. Let (X, M') be the pair associated with the integral points on V. Then $M \subset M'$ by definition. Thus if X satisfies M-approximation off T, then V satisfies strong approximation off T by Proposition 2.2.12. Conversely assume that V satisfies strong approximation off T.

Now we set $m = \max_{m_i<\infty}(m_i)$. If X is smooth then $m_i n_{\rho_i}, (m_i+1)n_{\rho_i} \in N_M$ for all $i \in \{1, \dots, n\}$ with $m_i < \infty$, so $n_{\rho_i} \in N_M$ and thus $N_M = N_{M'}$, so M-approximation holds if $T \neq \emptyset$ by Theorem 1.1.3. If furthermore $T = \emptyset$, then $N_{M'}^+ = N$ by Corollary 3.4.2. Thus every element $c \in N$ can be written as $c = \sum_{i=1}^n c_i n_{\rho_i}$ for some $c_i \geq 0$ with $c_i = 0$ if $m_i = \infty$, and therefore we have $m'c \in N_M^+$ for $m' \geq m$. Thus $c = (m+1)c + m(-c) \in N_M^+$, showing that (X, M) satisfies M-approximation off T if X is smooth.

If X is singular, then we consider a toric resolution of singularities $f\colon \tilde{X} \to X$. For every torus-invariant prime divisor $\tilde{D}$ with $f(\tilde{D}) \not\subset X \setminus V$, $\tilde{D} \not\subset f^{-1}D_i$ for any $i \in \{1, \dots, n\}$ with $m_i = \infty$. Denote the Zariski closure of $\tilde{D}$ in the toric integral model $\tilde{\mathcal{X}}$ of $\tilde{X}$ by $\tilde{\mathcal{D}}$. For each divisor D_i on X with $\tilde{D} \subset f^{-1}D_i$ as schemes, we have

$$n_v(\tilde{\mathcal{D}}, P) \leq n_v(f^{-1}\mathcal{D}_i, P) = n_v(\mathcal{D}_i, f(P))$$

for all $v \in \Omega_K^{<\infty}$ and $P \in \tilde{X}(K_v)$, where the equality is due to Proposition 2.1.31. This gives the inclusion $(\tilde{X}, \tilde{M}) \subset (\tilde{X}, f^{-1}M)$, where $\tilde{M}$ is the Campana condition for the divisor

$$\tilde{D}_{\tilde{\mathbf{m}}} = \sum_{\substack{i=1 \\ f(\tilde{D}_i)\not\subset X\setminus V}}^{\tilde{n}} \left(1 - \frac{1}{m}\right) \tilde{D}_i + \sum_{\substack{i=1 \\ f(\tilde{D}_i)\subset X\setminus V}}^{\tilde{n}} \tilde{D}_i,$$

where $\tilde{D}_1, \dots, \tilde{D}_{\tilde{m}}$ are the torus-invariant prime divisors on $\tilde{X}$. Since V satisfies strong approximation off T, $V \times_X \tilde{X} \subset \tilde{X}$ also satisfies strong approximation off T by Corollary 3.2.1. Since $\tilde{X}$ is smooth, the first part of the proof implies $(\tilde{X}, \tilde{M})$ satisfies $\tilde{M}$-approximation off T and thus $(\tilde{X}, f^{-1}M)$ satisfies $f^{-1}M$-approximation. Now Corollary 3.2.1 implies (X, M) satisfies M-approximation. □

3.5 Darmon points and root stacks

For smooth split toric varieties, we can generalize the connection between the fundamental group and strong approximation to M-approximation for Darmon points, using root stacks.

Definition 3.5.1. Let X be a scheme and let

$$D_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i,$$

for distinct prime Cartier divisors $D_1, \ldots, D_n$ on X and integers $m_1, \ldots, m_n \in \mathbb{N}^* \cup \{\infty\}$. Then We define *the root stack* associated to $(X, D_{\mathbf{m}})$ to be

$$(X, \sqrt[\mathbf{m}]{D}) := \left(X \setminus \bigcup_{\substack{i=1 \\ m_i=\infty}}^{n} D_i\right)_{\tilde{\mathbf{D}}, \tilde{\mathbf{m}}},$$

where the right hand side is as defined in [Cad07, Definition 2.2.4]. Here $\tilde{\mathbf{m}} = (m_i)_{i \in I}$ and $\tilde{\mathbf{D}} = (D_i)_{i \in I}$, where $I = \{i \in \{1, \ldots, n\} \mid m_i \neq \infty\}$.

The root stack $(X, \sqrt[\mathbf{m}]{D})$ is an algebraic stack which is Deligne-Mumford if $m_1, \ldots, m_n \in \mathcal{O}(X)^\times$ [Cad07, Theorem 2.3.3]. It comes with a morphism

$$(X, \sqrt[\mathbf{m}]{D}) \to X,$$

which is an isomorphism over U. For $i \in \{1, \ldots, n\}$ with $m_i < \infty$, the pullback of D_i along this morphism is $m_i \cdot \frac{1}{m_i} D_i$, where $\frac{1}{m_i} D_i$ is a prime Cartier divisor on $(X, D_{\mathbf{m}})$. Thus the morphism $(X, \sqrt[\mathbf{m}]{D}) \to X$ an isomorphism over U and it is ramified over D_i with multiplicity m_i for every $i \in \{1, \ldots, n\}$ with $m_i < \infty$. The following proposition illustrates the close relationship between root stacks and Darmon points.

Proposition 3.5.2. *Let (K, C) be a PF field, let $B \subset C$ be a nonempty open subset and let X be a proper variety over K with an integral model $\mathcal{X}$ over B. Let $\mathcal{D}_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) \mathcal{D}_i$ for $m_1, \ldots, m_n \in \mathbb{N}^* \cup \{\infty\}$ and prime Cartier divisors $\mathcal{D}_1, \ldots, \mathcal{D}_n$ on $\mathcal{X}$. Let $v \in B$, let $T \subset \Omega_K$ be a finite set of places, and let R be $\mathcal{O}_v$, $\mathbf{A}_K^T$, $\mathbf{A}_B^T$ or a field extension L of K. For every place $v \in B$, the Darmon points over R on $(\mathcal{X}, \mathcal{D}_{\mathbf{m}})$ as in Definition 2.1.19 are exactly the points $\mathcal{P} \in \mathcal{X}(R)$ such that there exists a factorisation*

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, dotted] \arrow[dr, "\mathcal{P}"'] & (\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}}) \arrow[d] \\ & \mathcal{X}. \end{tikzcd} \tag{3.5.1}$$

Moreover, if the divisors $\mathcal{D}_i$ pull back to Cartier divisors on $\operatorname{Spec} R$*, then the above factorisation is unique.*

In particular, if $\mathcal{P} \colon B \to \mathcal{X}$ is a morphism such that $\operatorname{Pic}(B) = 0$ *or* $\operatorname{im} \mathcal{P} \not\subset \bigcup_{i=1}^{n} \mathcal{D}_i$*, then $\mathcal{P}$ is a Darmon point on $(\mathcal{X}, \mathcal{D}_{\mathbf{m}})$ if and only if $\mathcal{P}$ factors through the root stack as in* (3.5.1).

Proof. For $i \in \{1, \ldots, n\}$, let L By the definition of $(\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}})$ and [Cad07, Remark 2.2.2, Remark 2.2.5], a morphism $\operatorname{Spec} R \to (\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}})$ is determined by a morphism $\mathcal{P} \colon \operatorname{Spec} R \to \mathcal{X} \setminus \bigcup_{\substack{i=1 \\ m_i=\infty}}^{n} \mathcal{D}_i$ together with isomorphism classes $(L_1, s_1), \ldots, (L_n, s_n)$ of line bundles on $\operatorname{Spec} R$ with a given global section and isomorphisms $\phi_i \colon L_i^{\otimes m_i} \to$

$\mathcal{P}^*\mathcal{O}(\mathcal{D}_i)$ such that $\phi_i(s_i^{m_i}) = \mathcal{P}^*1_{\mathcal{D}_i}$, where $1_{\mathcal{D}_i}$ denotes the canonical section of $\mathcal{O}(\mathcal{D}_i)$. Since $\operatorname{Pic}(R)$ is trivial, which for the adelic rings follows from Proposition 1.3.8, the line bundles $\mathcal{P}^*\mathcal{O}(\mathcal{D}_i)$ are trivial and thus the isomorphism classes just correspond to ideals on R. Thus the factorisation exists if and only if for all i with $m_i \neq \infty$ the ideal defining the closed subscheme $\mathcal{P} \cap \mathcal{D} \subset \operatorname{Spec} R$ is an m_i-th power of an ideal (which is uniquely determined by this property).

If the image of $\mathcal{P}$ is not contained in $\bigcup_{i=1}^n \mathcal{D}_i$, then the pullbacks $(\mathcal{P}^*\mathcal{D}_1, \mathcal{P}^*s_1), \dots, (\mathcal{P}^*\mathcal{D}_n, \mathcal{P}^*s_n)$ all exist as effective Cartier divisors and thus correspond to invertible ideals on R. For $i \in \{1, \dots, n\}$, any automorphism of $\mathcal{P}^*\mathcal{D}_i$ fixing $\mathcal{P}^*s_i$ corresponds to an automorphism of R-modules $R \to R$ fixing a nonzero divisor, and is therefore trivial. Thus the factorisation is necessarily unique.

Now consider a morphism $\mathcal{P}\colon B \to \mathcal{X}$. If $\operatorname{Pic}(B) = 0$, then by the same reasoning as above, $\mathcal{P}$ is a Darmon point on $(\mathcal{X}, \mathcal{D}_{\mathbf{m}})$ if and only if it factors through the root stack. If instead $\operatorname{im}\mathcal{P} \not\subset \bigcup_{i=1}^n \mathcal{D}_i$, then the divisors $\mathcal{D}_i$ pull back to effective Cartier divisors on B, and therefore [Cad07, Remark 2.2.2] implies that $\mathcal{P}$ is a Darmon point if and only if it factors though the root stack. □

The next proposition gives conditions for a root stack to be regular.

Proposition 3.5.3. *Let X be a regular scheme and let*

$$D_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i,$$

for distinct prime Cartier divisors $D_1, \dots, D_n$ on X and integers $m_1, \dots, m_n \in \mathbb{N}^* \cup \{\infty\}$ such that the support of $D_{\mathbf{m}}$ is an strict normal crossings divisor. Then $(X, \sqrt[\mathbf{m}]{D})$ is regular.

Proof. Without loss of generality, we can assume that $m_1, \dots, m_n$ are all finite, since a the restriction of a strict normal crossings divisor to an open is still strict normal crossings. As the statement is local, we can assume that $X = \operatorname{Spec} A$ is affine and $D_i = \operatorname{Spec} A/(s_i)$. By [Cad07, Example 2.4.1] this implies

$$(X, \sqrt[\mathbf{m}]{D}) \cong [\operatorname{Spec} R/(\mu_{m_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mu_{m_n})],$$

where $R = A[x_1, \dots, x_n]/(x_1^{m_1} - s_1, \dots, x_n^{m_n} - s_n)$ and μ_{m_i} acts trivially on A and x_j for $j \neq i$, and acts on x_i by $t_i \cdot x_i = t_i^{-1} x_i$. We will first prove that $\operatorname{Spec} R$ is regular. Since R is finite over A, for every maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$ with an extension to a maximal ideal $\mathfrak{m}' \in \operatorname{Spec} R$, the dimensions of the local rings agree: $\dim A_{\mathfrak{m}} = \dim R_{\mathfrak{m}'}$. Furthermore, since the support of $D_{\mathbf{m}}$ is a strict normal crossings divisor, the elements in $\{s_1, \dots, s_n\} \cap \mathfrak{m}$ are part of a regular system of parameters for $\mathfrak{m}$. and therefore the elements in $\{x_1, \dots, x_n\} \cap \mathfrak{m}'$ is part of a regular system of parameters for $\mathfrak{m}'$, since if I is the ideal generated by the $x_1, \dots, x_n$ contained in $\mathfrak{m}$ and I' is generated by the $s_1, \dots, s_n$ contained in $\mathfrak{m}'$, then $R_{\mathfrak{m}'}/I' \cong A_{\mathfrak{m}}/I$. Thus $R_{\mathfrak{m}'}$ is a regular local ring, so $\operatorname{Spec} R$ is regular.

Note that $\operatorname{Spec} R \to (X, \sqrt[\mathbf{m}]{D})$ is surjective, flat and of finite presentation. Consider a smooth cover $Y \to (X, \sqrt[\mathbf{m}]{D})$, where Y is a scheme. Then $\operatorname{Spec} R \times_{(X, \sqrt[\mathbf{m}]{D})} Y$ is a regular algebraic space since regularity is local in the smooth topology by [Stacks, Tag 036D]. Since $\operatorname{Spec} R \times_{(X, \sqrt[\mathbf{m}]{D})} Y \to Y$ is surjective, flat and of finite presentation, Y is regular by [Stacks, Tag 06QN] and thus $(X, \sqrt[\mathbf{m}]{D})$ is regular. □

Proposition 3.5.2 allows us to relate M-approximation with strong approximation on root stacks, as studied in [Chr20; San23b].

Definition 3.5.4. Let U be a stack over a PF field (K, C) and let $T \subset \Omega_K$ be a finite set of places. We say that U satisfies strong approximation off T if the map

$$U(K) \to U(\mathbf{A}_K^T)$$

has dense image, where the topology on $U(\mathbf{A}_K^T)$ is defined as in [Chr20, Definition 5.0.10].

Proposition 3.5.5. *Let (K, C) be a PF field, $T \subset \Omega_K$ a finite set of places, and X a smooth proper variety over K. Let $D_{\mathbf{m}} = \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) D_i$ for $m_1, \dots, m_n \in \mathbb{N}^* \cup \{\infty\}$ and smooth prime Cartier divisors $D_1, \dots, D_n$ on X. Assume that the support of $D_{\mathbf{m}}$ is an strict normal crossings divisor. Let (X, M) be the pair corresponding to the Darmon points on $(X, D_{\mathbf{m}})$. Then (X, M) satisfies M-approximation off T if and only if the root stack $(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation off T.*

Proof. Write $\tilde{X} = (X, \sqrt[\mathbf{m}]{D})$ and $\tilde{\mathcal{X}} = (\mathcal{X}, \sqrt[\mathbf{m}]{\mathcal{D}})$, for some integral model $\mathcal{X}$ over some open subset $B \subset C$ such that $\mathcal{D}_i$ is the closure of D_i in $\mathcal{X}$ and it is a prime Cartier divisor. By [Chr20, Proposition 13.0.2],

$$(X, \sqrt[\mathbf{m}]{D})(\mathbf{A}_K^T) = \prod_{v \in \Omega_K \setminus (B \cup T)} \tilde{X}(K_v) \times \prod_{v \in B \setminus T} \left(\tilde{X}(K_v), \tilde{\mathcal{X}}(\mathcal{O}_v)\right)$$

as topological spaces (the cited proposition is formulated for global fields and with $T = \emptyset$, but extends to this setting). By the assumptions on X and on $D_{\mathbf{m}}$, the root stack $(X, \sqrt[\mathbf{m}]{D})$ is geometrically regular, and thus smooth, by Proposition 3.5.3. This is because the divisors $D_1, \dots, D_n$ are smooth so the support of $D_{\mathbf{m}, \overline{K}}$ is a strict normal crossings divisor. Therefore by [Chr20, Proposition 7.0.8] there exists a scheme $\mathcal{Z}$ over B and a surjective smooth morphism $\pi \colon \mathcal{Z} \to \tilde{\mathcal{X}}$, such that $\pi(\mathcal{Z}(\mathcal{O}_v)) = \tilde{\mathcal{X}}(\mathcal{O}_v)$ for all $v \in B$ and $\pi(\mathcal{Z}(K_v)) = \tilde{X}(K_v)$ for any place $v \in \Omega_K$. In general [Chr20, Proposition 7.0.8] only gives a family of schemes $\mathcal{Z}_N$ with this property, but we can take the disjoint union of these to obtain $\mathcal{Z}$. Since $\tilde{X}$ is smooth over K, $\mathcal{Z}_K$ is smooth over K as well, and thus is locally a variety. Therefore Proposition 2.2.6 implies that $U(K_v) \subset \mathcal{Z}(K_v)$ is dense for any dense Zariski open subset $U \subset Z$ and any $v \in \Omega_K$. In particular this implies that $(X, M_{\text{fin}})(K_v) = (X \setminus \bigcup_{i=1}^n D_i)(K_v)$ lies dense in $\tilde{X}(K_v)$. Therefore, it follows that the image of $(X, M_{\text{fin}})(\mathbf{A}_K^T)$ is dense in $\tilde{X}(\mathbf{A}_K^T)$. Thus, by Proposition 2.2.5, $\tilde{X}(K)$ is dense in $\tilde{X}(\mathbf{A}_K^T)$ if and only if $(X, M)(K)$ is dense in $(X, M)(\mathbf{A}_K^T)$. □

We now consider the fundamental group $\pi_1(X, \sqrt[\mathbf{m}]{D})$ of the root stack as defined in [Noo04], which classifies étale covers $f \colon Y \to (X, \sqrt[\mathbf{m}]{D})$, where Y is an algebraic stack.

Definition 3.5.6. A morphism $f \colon Y \to X$ of connected algebraic stacks is an *étale cover* if it is a finite étale morphism.

Note that such a morphism is always representable, see for example [Stacks, Tag 0CHT], so étale covers coincide with what Noohi calls covering maps.

Similarly to the fundamental group of schemes, different choices of a base point yield the same fundamental group up to isomorphism [Noo04, page 9], so we will leave the choice of the point implicit. The following lemma classifies the étale covers of root stacks in terms of the coarse spaces, and thus gives a concrete description of its étale fundamental group.

Lemma 3.5.7. *Let X be a connected locally Noetherian scheme and let $D_1, \dots, D_n$ be distinct prime Cartier divisors on X, and $m_1, \dots, m_n \in \mathbb{N}^*$ whose images in $\mathcal{O}(X)$ are invertible. Assume that the support of*

$$D_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i,$$

is a strict normal crossings divisor.

Then there is a one-to-one correspondence between

1. *Étale covers $\tilde{f}\colon \tilde{Y} \to (X, \sqrt[\mathbf{m}]{D})$, and*
2. *Finite morphisms of connected schemes $f\colon Y \to X$, such that f is étale over $X \setminus \bigcup_{i=1}^{n} D_i$, and for all $i \in \{1, \dots, n\}$ the pullback satisfies $f^*D_i = \sum_{\beta \in \mathcal{B}_i} e_{i,\beta} \tilde{D}_{i,\beta}$ for distinct prime Cartier divisors $\tilde{D}_{i,\beta}$ on Y such that $e_{i,\beta} | m_i$ for all $\beta \in \mathcal{B}_i$.*

Proof. Recall that our schemes are by assumption separated so since $m_1, \dots, m_n$ are invertible in $\mathcal{O}(X)$, the stack $(X, \sqrt[\mathbf{m}]{D})$ is separated and Deligne-Mumford by [Cad07, Corollary 2.3.4]. Therefore for any étale cover $\tilde{Y} \to (X, \sqrt[\mathbf{m}]{D})$, $\tilde{Y}$ is separated as the map is finite and hence affine [Stacks, Tag 01S7] and Deligne-Mumford since it is étale [Stacks, Tag 0CIQ]. Therefore by the Keel-Mori Theorem [Ryd13, page 631], there exists a coarse moduli space $\tilde{Y} \to Y$, where Y is an algebraic space. Since the coarse moduli space is universal for maps to algebraic spaces by definition [Ryd13, Definition 6.8], the cover $\tilde{f}$ descends to a morphism $f\colon Y \to X$. We thus obtain a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{f}"] \arrow[d] & (X, \sqrt[\mathbf{m}]{D}) \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd} \tag{3.5.2}$$

We will show that Y is a scheme and that $f\colon Y \to X$ satisfies the desired properties.

Since the statement is local on X, we can reduce to the case that $X = \operatorname{Spec} A$ is affine and the $D_i = \operatorname{Spec} A/(s_i)$. By [Cad07, Example 2.4.1] this implies

$$(X, \sqrt[\mathbf{m}]{D}) \cong [\operatorname{Spec} R/(\mu_{m_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mu_{m_n})],$$

where $R = A[t_1, \dots, t_n]/(x_1^{m_1} - s_1, \dots, x_n^{m_n} - s_n)$ and μ_{m_i} acts trivially on A and x_j for $j \neq i$, and acts on x_i by $t_i \cdot x_i = t_i^{-1} x_i$. Write $Z = \operatorname{Spec} R$. Note that since R is a finite A-algebra, the map $Z \to X$ is finite.

We now prove that $f\colon Y \to X$ is a finite morphism. Let $U \to Y$ be a finite étale morphism from a scheme U and consider the following commutative diagram consisting of Cartesian squares:

$$\begin{tikzcd}
U \times_X Z \arrow[r] \arrow[d] & \tilde{Y} \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\
U \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[r, "\tilde{f}"] \arrow[d] & (X, \sqrt[\mathbf{m}]{D}) \arrow[d] \\
U \arrow[r] & Y \arrow[r, "f"] & X.
\end{tikzcd}$$

Since $U \times_X Z \to Z$ and $Z \to X$ are finite morphisms of schemes, $U \times_X Z \to X$ is a finite morphism as well.

Since X is locally Noetherian, [Ryd13, Theorem 6.12] implies that the map f is proper and thus locally of finite type. Therefore $U \to X$ is also locally of finite type, and since $U \times_X Z \to U$ is surjective, [Stacks, Tag 0GWS] implies that $U \to X$ is quasi-finite, and by [Stacks, Tag 02LS] it is finite and hence f is finite as well. By [Stacks, Tag 03XX] Y is a scheme.

Since $(X, \sqrt[\mathbf{m}]{D}) \to X$ is étale outside the support of $D_{\mathbf{m}}$, f is étale outside its support as well. Furthermore, since $\tilde{f}$ is étale and the pullback of D_i along the morphism $(X, \sqrt[\mathbf{m}]{D}) \to X$ is $m_i E_i$ for a prime divisor E_i, the multiplicity of every prime divisor appearing in the pullback of D_i to $\tilde{Y}$ is exactly m_i. By the commutativity of the diagram (3.5.2), the multiplicities $e_{i,\beta}$ divide m_i.

Conversely, for any such morphism $f\colon Y \to X$ satisfying the properties in (2), we will show we can construct a root stack $\tilde{Y}$ over Y and an étale cover $\tilde{f}\colon \tilde{Y} \to (X, \sqrt[\mathbf{m}]{D})$ inducing the morphism f on coarse spaces. Since the support $\sup(D_{\mathbf{m}})$ is strict normal crossings, Proposition 3.5.3 implies all points $\tilde{x} \in (X, \sqrt[\mathbf{m}]{D}) \times_X \sup(D_{\mathbf{m}})$ are regular. Since the statement to be proved is local on X, and the regular locus is open, we can assume that $(X, \sqrt[\mathbf{m}]{D})$ is regular. Take $\tilde{Y}$ to be the root stack $\tilde{Y} = (Y, \sqrt[\tilde{\mathbf{m}}]{\tilde{D}})$, where $\tilde{D}_{\tilde{\mathbf{m}}} = \sum_{i=1}^{n} \sum_{\beta \in \mathcal{B}_i} \left(1 - \frac{e_{i,\beta}}{m_i} \tilde{D}_{i,\beta}\right)$. Since f is a finite map of schemes, $\tilde{Y} \times_X Z \to Z$ is finite, so $\tilde{Y} \times_X Z$ is a scheme by [Stacks, Tag 03XX]. The morphism $\tilde{Y} \times_X Z \to Z$ is étale at the codimension 1 points appearing in the pullback of D_i to $\tilde{Y} \times_X Z$, for every $i \in \{1, \ldots, n\}$. Therefore purity of the ramification locus [Stacks, Tag 0EA4] implies that $\tilde{Y} \times_X Z \to Z$ is étale at every point above $\bigcup_{i=1}^{n} D_i$, and thus $\tilde{Y} \times_X Z \to Z$ is étale everywhere. Since $Z \to (X, \sqrt[\mathbf{m}]{D})$ is an étale cover, this implies $\tilde{f}\colon \tilde{Y} \to (X, \sqrt[\mathbf{m}]{D})$ is an étale cover. □

Now we can explicitly compute the fundamental group of a toric root stack. For an abelian group G, let $\widehat{G} = \varprojlim_H G/H$ be the profinite completion, where H runs over all normal subgroups of G with finite index in G. Similarly, for a prime number p, we let

$$\widehat{G}^{(p)} = \varprojlim_{H,\, p \nmid [G:N]} G/H$$

be the prime-to-p completion. For a stack Z, we write $\pi_{1,p\nmid}(Z)$ for the quotient of $\pi_1(Z)$ corresponding to covers with degree coprime to p.

Lemma 3.5.8. *Let X be a smooth toric variety over an algebraically closed field $\overline{K}$ and let $D_1, \ldots, D_n$ be the the torus-invariant prime divisors on X and let $m_1, \ldots, m_n \in \mathbb{N}^* \cup \{\infty\}$ whose images in $\mathcal{O}(X)$ are invertible and let $D_{\mathbf{m}}$ be the* *corresponding Campana divisor. Let (X, M) be the pair corresponding to the Darmon points on $(X, D_{\mathbf{m}})$. If* $\operatorname{char}(\overline{K}) = 0$*, then*

$$\pi_1(X, \sqrt[\mathbf{m}]{D}) \cong \widehat{N/N_M},$$

while if $\overline{K}$ has characteristic $p > 0$ then

$$\pi_{1,\,p\nmid}(X, \sqrt[\mathbf{m}]{D}) \cong \widehat{N/N_M}^{(p)}.$$

Proof. We only write the proof for characteristic 0, as the positive characteristic case is analogous. We write Σ for the fan of $X \setminus \lfloor D_{\mathbf{m}} \rfloor$. If $X \setminus \lfloor D_{\mathbf{m}} \rfloor$ is just the dense open torus (so if $m_1 = \cdots = m_n = \infty$) then the statement is true by [BS13, Propostion 1.1]. By this we see that a Galois cover of $\mathbb{G}_m^d$ is necessarily just a pair of d coverings of $\mathbb{G}_m$. Therefore, we can without loss of generality assume that $X \setminus \lfloor D_{\mathbf{m}} \rfloor$ does not have torus factors. By [AP13, Theorem 4] any finite morphism $f\colon Y \to X \setminus \lfloor D_{\mathbf{m}} \rfloor$ of connected schemes, such that the degree is coprime to p, is a morphism of toric varieties. The cited theorem is only stated for complete toric varieties, but the proof also works for the toric varieties without torus factors. Furthermore [AP13, Definition 1, Lemma 1] imply that the étale covers $f\colon Y \to (X, \sqrt[\mathbf{m}]{D})$ exactly correspond to maps of fans $(N', \Sigma) \to (N, \Sigma)$, where $N' \subset N$ is a sublattice of finite index containing N_M. Furthermore, such a cover is a N/N'-cover. By letting N' run over all lattices $N' \subset N$ of finite index which contain N_M we obtain the result. □

Proof of Corollary 1.1.10. Combine Theorem 1.1.3 with Lemma 3.5.8. For the second part, note that the coarse space of $(X, \sqrt[\mathbf{m}]{D})$ is $X \setminus \lfloor D_{\mathbf{m}} \rfloor$ and use Proposition 3.4.1. □

Remark 3.5.9. Corollary 1.1.10 extends to characteristic p to give a necessary condition for strong approximation on a toric root stack with invertible multiplicities. If $(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation off T, then $|\pi_{1,p\nmid}(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}})| \in \rho(K, C)$ and similarly strong approximation implies that $(X_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}})$ does not have étale covers of degree coprime to p. However, this is not a sufficient condition, since the quotient group N/N_M corresponding to Darmon points may have p-torsion.

Example 3.5.10. If $\operatorname{char}(K) = 0$, $X = \mathbb{P}^1$ and $D_{\mathbf{m}} = \frac{1}{2}(0) + \frac{1}{2}(\infty)$ then the morphism $\mathbb{P}^1_K \to \mathbb{P}^1_K$ given by $(x_0 : x_1) \mapsto (x_0^2 : x_1^2)$ factors as the étale morphism $\mathbb{P}^1_K \to (\mathbb{P}^1_K, \sqrt[\mathbf{m}]{D})$ followed by the coarse moduli space morphism $(\mathbb{P}^1, \sqrt[\mathbf{m}]{D}) \to \mathbb{P}^1$. The étale morphism corresponds to the nontrivial element in $\pi_1(\mathbb{P}^1_{\overline{K}}, \sqrt[\mathbf{m}]{D_{\overline{K}}}) \cong \mathbb{Z}/2\mathbb{Z}$.

More generally, if $(\mathbb{P}^n, M)$ is a toric pair corresponding to Darmon points, then the group N/N_M is nontrivial as soon as two multiplicities are not coprime. Therefore we obtain the following consequence of Corollary 1.1.10.

Corollary 3.5.11. *Let (K, C) be a PF field and let $T \subset \Omega_K$ be a finite nonempty set of places. Let $D_{\mathbf{m}}$ be the $\mathbb{Q}$-divisor on $\mathbb{P}^{n-1}_K$ given by*

$$D_{\mathbf{m}} = \sum_{i=0}^{n-1} \left(1 - \frac{1}{m_i}\right) D_i, \quad D_i = \{x_i = 0\}.$$

Then $(\mathbb{P}^{n-1}_K, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation off T if $\gcd(m_i, m_j) \in \rho(K, C)$ *for every $i \neq j$. The converse also holds if* $\operatorname{Pic}(C)$ *is finitely generated. Furthermore,*

$(\mathbb{P}^{n-1}_K, \sqrt[\mathbf{m}]{D})$ *satisfies strong approximation if and only if* $m_i < \infty$ *for all* $i \in \{0, \ldots, n\}$ *and* $\gcd(m_i, m_j) = 1$ *for every* $i \neq j$.

Proof. Let $(\mathbb{P}^{n-1}_K, M)$ be the pair corresponding to the Darmon points on $(\mathbb{P}^{n-1}_K, D_{\mathbf{m}})$. Consider the matrix

$$\begin{pmatrix} -m_0 & m_1 & 0 & \ldots & 0 \\ -m_0 & 0 & m_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_0 & 0 & 0 & \ldots & m_{n-1} \end{pmatrix},$$

where the columns correspond to generators of N_M (if $m_i = \infty$, we make the corresponding column 0 instead). By Theorem 1.1.3, M-approximation off T is satisfied if and only if the matrix has full rank and induces surjective homomorphisms $(\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^{n-1}$ for every prime number $p \notin \rho(K, C)$. The matrix having full rank is equivalent to $m_i = \infty$ for at most one $i \in \{0, \ldots, n-1\}$. The surjectivity at a prime number $p \notin \rho(K, C)$ is equivalent to some $n \times n$ minor of the matrix not being divisible by p. As the maximal minors are, up to sign, of the form $\prod_{\substack{i=0 \\ i \neq j}}^n m_i$ for some $0 \leq j \leq n$, this is equivalent to $\gcd(\prod_{\substack{i=0 \\ i \neq 0}}^n m_i, \ldots, \prod_{\substack{i=0 \\ i \neq n}}^n m_i) \in \rho(K, C)$. This is in turn equivalent to $\gcd(m_i, m_j) \in \rho(K, C)$ for every $i \neq j$. The proof for $T = \emptyset$ is analogous. $\square$

The formula obtained above generalizes to a simple sufficient criterion for strong approximation on toric stacks.

Corollary 3.5.12. *Let K be a PF field and let $T \subset \Omega_K$ be a nonempty set of places. Let X be a smooth split toric variety and let $(X, \sqrt[\mathbf{m}]{D})$ be the root stack corresponding to $D_{\mathbf{m}} = \sum_{i=0}^n \left(1 - \frac{1}{m_i}\right) D_i$, and let $P_\sigma(x_1, \ldots, x_n) = \prod_{\substack{i=1 \\ \rho_i \subset \sigma}}^n x_i$ for a maximal cone σ. If $\gcd_{\sigma \in \Sigma_{max}}(P_\sigma(m_1, \ldots, m_n)) \in \rho(K, C)$, then $(X, \sqrt[\mathbf{m}]{D})$ satisfies strong approximation off T. Here Σ_{max} is the set of maximal cones in the fan of X.*

Proof. If $\mathcal{V}_\sigma \cong \mathbb{A}^n$ is the affine open in the toric integral model $\mathcal{X}$ corresponding to the maximal cone σ, then for any $v \in \Omega_K^{<\infty}$, the lattice N_σ spanned by the image of $(\mathcal{X}, \mathcal{M})(\mathcal{O}_v) \cap \mathcal{V}_\sigma(\mathcal{O}_v)$ under ϕ_v in N_M is generated by $m_i n_{\rho_i}$ for $\rho_i \subset \sigma$. Thus $|N : N_\sigma| = P_\sigma(m_1, \ldots, m_n)$. Since $N_\sigma \subset N_M$, this implies $|N : N_M|$ divides $P_\sigma(m_1, \ldots, m_n)$, giving the result. $\square$

While Corollary 3.5.12 is a sufficient criterion, it is not a necessary condition on many toric varieties, such as Hirzebruch surfaces.

Example 3.5.13 (Hirzebruch surfaces)**.** Let (K, C) be a PF field such that $\mathrm{Pic}(C)$ is finitely generated, $T \subset \Omega_K$ a nonempty set of places and $r \geq 0$ an integer. Consider the Hirzebruch surface H_r given by the fan with ray generators $n_{\rho_1} = (-1, r), n_{\rho_2} = (0, 1), n_{\rho_3} = (1, 0), n_{\rho_4} = (0, -1)$ and choose corresponding multiplicities m_1, m_2, m_3, m_4. If (H_r, M) is the pair for the Darmon points with these multiplicities, then by looking at the generators modulo a prime number we see that the prime numbers dividing $|N : N_M|$ are the prime numbers dividing

$$\gcd(m_1 m_2, m_1 m_4, m_2 m_3, m_3 m_4, r m_1 m_3).$$

By Theorem 1.1.3, the pair (H_r, M) satisfies M-approximation off T if and only if $|N : N_M| \in \rho(K, C)$. Moreover, (H_r, M) satisfies M-approximation if and only if $N = N_M$, $m_1, m_3, m_4 < \infty$ and $(r, m_2) \neq (0, \infty)$. Note the additional rm_1m_3 factor showing up in the index of $|N : N_M|$ compared to the criterion given in Corollary 3.5.12, showing that M-approximation can hold even if the condition in the corollary is violated.

Finally we also consider M-approximation for pairs corresponding to Darmon points on a singular variety.

Example 3.5.14. Let (K, C) be a PF field such that $\mathrm{Pic}(C)$ is finitely generated, $T \subset \Omega_K$ a nonempty set of places and $r \geq 1$ an integer. Consider the weighted projective plane $\mathbb{P}_K(1, 1, r)$ for $r \geq 1$ with rays generated by $n_{\rho_0} = (-1, r), n_{\rho_1} = (1, 0), n_{\rho_2} = (0, -1)$ and choose corresponding multiplicities m_0, m_1, m_2 with $m_0, m_1 < \infty$. Then the Darmon points on $\mathbb{P}_K(1, 1, r)$ satisfy M-approximation off T if and only if $\gcd(m_0, m_1), \gcd(m_0m_1, m_2, r - 1) \in \rho(K, C)$. Furthermore the Darmon points on $\mathbb{P}_K(1, 1, r)$ satisfy M-approximation if and only if $\gcd(m_0, m_1) = \gcd(m_0m_1, m_2, r - 1) = 1$ and $m_2 < \infty$. In particular, if $r = 2$, then whether M-approximation is satisfied (off T or off $\emptyset$) does not depend on the value of m_2.

4. $\mathcal{M}$-points of bounded height

In this chapter we propose an asymptotic formula for the number of $\mathcal{M}$-points of bounded height, which generalizes Manin's conjecture for rational points of bounded height, as well as its extension [PSTVA21, Conjecture 1.1] for Campana points. In Chapter 5, we will then prove this asymptotic formula for toric pairs over $\mathbb{Q}$.

4.1 Divisors on pairs

In the remaining chapters, we restrict our attention to smooth pairs $(X, M) = (X, ((D_i)_{i\in\{1,\dots,n\}}, \mathfrak{M}))$.

Definition 4.1.1. A pair (X, M) over a field K of characteristic 0 is called *smooth* if

1. X is a smooth variety over K,
2. every divisor D_i is connected, nonempty and smooth over K, $\sum_{i=1}^n D_i$ is a strict normal crossings divisor as defined in [Stacks, Tag 0BI9],
3. the monoid $\mathfrak{M}_{\mathrm{mon}} \subset \mathbb{N}^n$ generated by $\mathfrak{M} \cap \mathbb{N}^n$ is finitely generated.

The use of the term "smooth" is motivated by the fact that strict normal crossings pairs as considered in logarithmic geometry are log smooth, see e.g. [Ogu18, Chapter IV, Example 3.1.14]. Furthermore, Proposition 3.5.3 shows that a root stack $(X, \sqrt[\mathfrak{m}]{D})$ corresponding to a smooth pair (X, M) is smooth.

Assumption 4.1.2. From now on, all pairs considered will be smooth. Furthermore, we will always assume that X is a connected, proper variety.

Definition 4.1.3. A smooth pair (X, M) is *proper* if X is proper and $\mathfrak{M}$ contains a non-zero multiple of the standard basis vector $\mathbf{e}_i$ for all $i \in \{1, \dots, n\}$.

If (X, M) is a pair corresponding to Darmon points, then the corresponding root stack $(X, \sqrt[\mathfrak{m}]{D})$ lies over X and $(X, \sqrt[\mathfrak{m}]{D}) \to X$ is a ramified morphism which is an isomorphism over the open set $U = X \setminus (D_1 \cup \dots \cup D_n)$.

More generally we can regard (X, M) as some sort of geometric space akin to a scheme or stack lying above X. If $\mathfrak{M} \cap \mathbb{N}^n$ is a monoid with topological closure $\mathfrak{M}$, then (X, M) determines a functor $S \mapsto (X, M)(S)$ as we have seen in Remark 2.1.18. We view the natural inclusion map $(X, M) \to X$ as analogous to an "generically finite morphism" which is an isomorphism over U. Inspired by this view we introduce corresponding objects to the pair (X, M), such as divisors and a Picard group.

Notation 4.1.4. For a smooth pair (X, M), we write

$$\Gamma_M = \{\mathbf{m} \in \mathfrak{M} \mid \mathbf{m} \text{ is not a sum of two nonzero elements in } \mathfrak{M}_{\text{red,mon}}\}$$

for the unique minimal set of generators of the monoid $\mathfrak{M}_{\text{red,mon}}$. Since we assume that (X, M) is smooth, Γ_M is always finite. For $\mathbf{m} \in \mathbb{N}^n$, we write $\mathcal{C}_{\mathbf{m}}$ for the finite set of connected components of the set

$$\bigcap_{\substack{i=1 \\ m_i > 0}}^{n} D_i \subset X.$$

Note that by [Stacks, Tag 0BIA], every such connected component is irreducible. Furthermore, we will write $\Gamma_{M,\mathcal{C}}$ for the set of tuples $(\mathbf{m}, c)$ with $\mathbf{m} \in \Gamma_M$ and $c \in \mathcal{C}_{\mathbf{m}}$. Similarly, we also write $\mathbb{N}^n_{\mathcal{C}}$ for the set of tuples $(\mathbf{m}, c)$ with $\mathbf{m} \in \mathbb{N}^n$ and $c \in \mathcal{C}_{\mathbf{m}}$. If for $\mathbf{m}$ there is a unique component c, we will routinely identify $\mathbf{m}$ and $(\mathbf{m}, c)$.

Example 4.1.5. If $\mathfrak{M}$ only contains the element $(0, \ldots, 0)$ then Γ_M is the empty set.

Example 4.1.6. If (X, M) is the pair corresponding to the Darmon points for the Campana pair $(X, D_{\mathbf{m}})$, where $D_{\mathbf{m}} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i$ for positive integers $m_1, \ldots, m_n$, then $\Gamma_M = \{m_1\mathbf{e}_1, \ldots, m_n\mathbf{e}_n\}$. If (X, M') is the pair corresponding to the Campana points for the Campana pair $(X, D_{\mathbf{m}})$, then $\Gamma_{M'} = \{m_1\mathbf{e}_1, \ldots, (2m_1 - 1)\mathbf{e}_1, m_2\mathbf{e}_2, \ldots, (2m_n - 1)\mathbf{e}_n\}$.

To each element $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$ we associate a formal symbol $\tilde{D}_{\mathbf{m},c}$, which we will refer to as a prime divisor on (X, M).

Definition 4.1.7. The *group of divisors on* (X, M) is

$$\operatorname{Div}(X, M) = \operatorname{Div}(U) \times \bigoplus_{(\mathbf{m},c) \in \Gamma_{M,\mathcal{C}}} \mathbb{Z}(\tilde{D}_{\mathbf{m},c}),$$

where $\operatorname{Div}(U)$ is the group of divisors on U. A divisor on (X, M) is an element of $\operatorname{Div}(X, M)$. Similarly, a $\mathbb{Q}$-divisor is an element of $\operatorname{Div}(X, M)_{\mathbb{Q}}$ and analogously, an $\mathbb{R}$-divisor is an element of $\operatorname{Div}(X, M)_{\mathbb{R}}$. A *prime divisor* $\tilde{D}$ on (X, M) is a prime divisor on U or $\tilde{D} = \tilde{D}_{\mathbf{m},c}$ for some $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$. Finally, a $\mathbb{Q}$-divisor is called *effective* if it is a nonnegative $\mathbb{Q}$-linear combination of prime divisors on (X, M).

Notation 4.1.8. If D, D' are two $\mathbb{Q}$-divisors on a pair (X, M), then we write $D \geq D'$ if $D - D'$ is effective.

In order to define the Picard group of a pair, we first need to define when two divisors on a pair are linearly equivalent. We will define this notion by introducing the pullback $\operatorname{pr}_M^* \colon \operatorname{Div}(X) \to \operatorname{Div}(X, M)$ from divisors on X to divisors on (X, M). In order to define this homomorphism, we first need to determine for every $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$ and every divisor $D \in \operatorname{Div}(X)$ what the coefficient $\mu((\mathbf{m}, c), D)$ of $\tilde{D}_{\mathbf{m},c}$ in the pullback of D should be.

Let $\mathcal{O}_{X,c}$ be the local ring at the closed subscheme $c \subset X$. Let $f_1, \ldots, f_n \in \mathcal{O}_{X,c}$ be local equations of the divisors $D_1, \ldots, D_n$, such that $f_i = 1$ for all $i \in$

$\{1, \ldots, n\}$ satisfying $c \not\subset D_i$. We consider the ring $R_{\mathbf{m},c} = \mathcal{O}_{X,c}[X_1, \ldots, X_n]/(X_1^{m_1} - f_1, \ldots, X_n^{m_n} - f_n)$ obtained by adjoining $f_i^{1/m_i} := \overline{X_i}$ to $\mathcal{O}_{X,c}$ for all $i \in \{1, \ldots, n\}$ with $m_i > 0$. By the strict normal crossings assumption on the divisors, the proof of Proposition 3.5.3 shows that $R_{(\mathbf{m},c)}$ is a regular local ring whose maximal ideal I is generated by all elements f_i^{1/m_i} with $m_i > 0$.

Definition 4.1.9. Let $(\mathbf{m}, c) \in \mathbb{N}^n_{\mathcal{C}}$ and let $D \in \operatorname{Div}(X)$ be an effective divisor given by a local equation $g = 0$, where $g \in \mathcal{O}_{X,c}$. Then we define

$$\mu((\mathbf{m}, c), D) := \max\{\mu \in \mathbb{N} \mid g \in I^\mu\},$$

where the ideal $I \subset R_{(\mathbf{m},c)}$ is defined as above. More generally, if $D = D_1 - D_2$ is a difference of two effective divisors on X with disjoint support, we set

$$\mu((\mathbf{m}, c), D) = \mu((\mathbf{m}, c), D_1) - \mu((\mathbf{m}, c), D_2).$$

The function $\mu((\mathbf{m}, c), -)$ is a group homomorphism, as the next proposition shows.

Proposition 4.1.10. *Let R be a regular local ring with a nonzero maximal ideal I. Then the function $v\colon R \setminus \{0\} \to \mathbb{N}$ given by*

$$v(g) = \max\{r \in \mathbb{N} \mid g \in I^r\}$$

for $g \in R$ extends to a valuation on the fraction field K of R. In particular $v(gg') = v(g) + v(g')$ for any $g, g' \in R$.

Consequently,

$$\mu((\mathbf{m}, c), -)\colon \operatorname{Div}(X) \to \mathbb{Z}$$

is a group homomorphism.

Proof. Let n be the Krull dimension of R and let $\{f'_1, \ldots, f'_n\}$ be a minimal set of generators of I. Then by adjoining the elements $f'_i/f'_j \in K$ to R for $i, j \in \{1, \ldots, n\}$, we obtain a local ring $R' \supset R$. The extension $I' \subset R'$ of I is a principal maximal ideal, so R' is a discrete valuation ring. The function v is now simply the restriction of the valuation on K determined by R' since $I = I' \cap R$. $\square$

The function μ is particularly easy to compute on the divisors $D_1, \ldots, D_n$.

Remark 4.1.11. For the divisors $D_1, \ldots, D_n$, the definition of $\mu((\mathbf{m}, c), D_i)$ simply reduces to

$$\mu((\mathbf{m}, c), D_i) = m_i.$$

Now we are ready to define the pullback.

Definition 4.1.12. Let the *pullback* of divisors on X to divisors on (X, M) be the group homomorphism

$$\operatorname{pr}_M^*\colon \operatorname{Div}(X) \to \operatorname{Div}(X, M)$$

defined by

$$D_i \mapsto \sum_{(\mathbf{m},c) \in \Gamma_{M,\mathcal{C}}} m_i \tilde{D}_{\mathbf{m},c},$$

for $i = 1, \dots, n$ and

$$D \mapsto D + \sum_{(\mathbf{m},c)\in\Gamma_{M,\mathcal{C}}} \mu((\mathbf{m}, c), D)\tilde{D}_{\mathbf{m},c}$$

for every divisor $D \in \mathrm{Div}(U)$.

Remark 4.1.13. Let $(\mathbf{m}, c) \in \mathbb{N}^n_{\mathcal{C}}$. If c is not contained in the support of a divisor D on X, then

$$\mu((\mathbf{m}, c), D) = 0.$$

Remark 4.1.14. The integer $\mu((\mathbf{m}, c), D)$ is the largest integer μ such that there is an inclusion of subschemes

$$\bigcap_{\substack{i=1\\ m_i>0}}^{n} \frac{\mu \operatorname{lcm}(m_1, \dots, m_n)}{m_i} D_i \subset \operatorname{lcm}(m_1, \dots, m_n)D$$

in an open neighbourhood of the generic point of c, where the intersection is the scheme theoretic intersection. The role of the least common multiple $\operatorname{lcm}(m_1, \dots, m_n)$ here is to ensure that the left hand side is a well defined closed subscheme of X.

Remark 4.1.15. The pullback of divisors to (X, M) is intimately related to the theory of weighted (stacky) blow-ups as described in [QR22]. For $(\mathbf{m}, c) \in \mathbb{N}^n_{\mathcal{C}}$, consider an open subset $X' \subset X$ such that the divisors $D_1, \dots, D_n$ are principal on X' and $\tilde{D}_{\mathbf{m}} = \tilde{D}_{\mathbf{m},c}$. Let $I_\bullet$ be the graded $\mathcal{O}_{X'}$-subalgebra of $\mathcal{O}_{X'}[t]$ generated by $f_1 t^{m_1}, \dots, f_n t^{m_n}$, where $f_i = 0$ is a local equation for D_i. If $\pi \colon \mathrm{Bl}_{I_\bullet} X' \to X'$ is the corresponding weighted blow-up as defined in [QR22, Definition 3.2.1] and $D \in \mathrm{Div}(X')$, then the coefficient of the exceptional divisor in $\pi^* D$ is equal to $\mu((\mathbf{m}, c), D)$.

Note that $\mathrm{Div}(U)$ embeds into $\mathrm{Div}(X, M)$ in two distinct ways. The first is by the embedding $\mathrm{Div}(U) \hookrightarrow \mathrm{Div}(X, M) \colon D \mapsto D$ directly given by the definition of $\mathrm{Div}(X, M)$. The second one is given by composing $\mathrm{Div}(U) \hookrightarrow \mathrm{Div}(X)$ with the pullback $\mathrm{Div}(X) \to \mathrm{Div}(X, M)$ to obtain the map $D \mapsto \mathrm{pr}_M^* D$.

Definition 4.1.16. For a divisor $D \in \mathrm{Div}(U)$, we call $D \in \mathrm{Div}(X, M)$ the *strict transform* of D in (X, M).

This terminology is motivated by the fact that $\mathrm{pr}_M^* D - D$ is a divisor whose restriction to U is trivial. This is illustrated by the following example.

Example 4.1.17. Let (X, M) be a pair obtained by taking X to be a smooth variety, the divisors D_1, D_2 to be two smooth divisors on X intersecting transversally with connected intersection $D_1 \cap D_2$, and let $\mathfrak{M} \subset \mathbb{N}^2$ be the monoid generated by $(1, 1)$. Let $\tilde{X} = \mathrm{Bl}_{D_1\cap D_2} X \setminus (\tilde{D}_1 \cup \tilde{D}_2)$, where $\tilde{D}_1, \tilde{D}_2$ are the strict transforms of D_1 and D_2 respectively. Then the homomorphism $\mathrm{Div}(X, M) \to \mathrm{Div}(\tilde{X})$, given by sending $\mathrm{Div}(U)$ to itself and $\tilde{D}_{(1,1)}$ to the exceptional divisor, is an isomorphism respecting the pullbacks of divisors in U. Thus under this isomorphism, the strict transform of a divisor in U in $\mathrm{Div}(\tilde{X})$ corresponds to the strict transform in $\mathrm{Div}(X, M)$.

While the function $\mu \colon \mathbb{N}^n_{\mathcal{C}} \times \mathrm{Div}(X) \to \mathbb{Z}$ is linear in the divisor argument, it is not linear in the first argument as the next examples show.

Example 4.1.18. Let $X = \mathbb{P}^2$, $n = 2$, $D_1 = \{X_1 = 0\}$, $D_2 = \{X_2 = 0\}$, $\mathfrak{M} = \mathbb{N}^2 \setminus \{(1,0),(0,1)\}$. Then $D_1 \cap D_2 = \{(0:0:1)\}$, and the divisor $D = \{X_1 = X_2\}$ has pullback $\mathrm{pr}_M^* D = D + \tilde{D}_{(1,1)}$. In particular,

$$1 = \mu((1,1),D) \geq \mu((1,0),D) + \mu((0,1),D) = 0$$

For an example with no multiplicities equal to zero, we can take

$$3 = \mu((3,3),D) \geq \mu((2,1),D) + \mu((1,2),D) = 2.$$

However, μ is a *concave* function in the second argument, as the next lemma shows. We will exploit this fact in Lemma 4.2.16 to understand the Fujita invariant and the b-invariant of line bundles.

Lemma 4.1.19. *Let (X,M) be a smooth pair. Then for all $(\mathbf{m},c),(\mathbf{m}',c') \in \mathbb{N}^n_{\mathcal{C}}$ and for every effective divisor D on X,*

$$\mu((\mathbf{m}+\mathbf{m}',\tilde{c}),D) \geq \mu((\mathbf{m},c),D) + \mu((\mathbf{m}',c'),D), \tag{4.1.1}$$

for every connected component $\tilde{c} \in \mathcal{C}_{\mathbf{m}+\mathbf{m}'}$ of $c \cap c'$. Furthermore, for any $t \in \mathbb{N}^*$

$$\mu((t\mathbf{m},c),D) = t\mu((\mathbf{m},c),D). \tag{4.1.2}$$

Therefore, for all $\lambda,\lambda' \in \mathbb{Q}$ with $\lambda\mathbf{m} \in \mathbb{N}^n$ and $\lambda\mathbf{m}' \in \mathbb{N}^n$,

$$\mu((\lambda\mathbf{m}+\lambda'\mathbf{m}',\tilde{c}),D) \geq \lambda\mu((\mathbf{m},c),D) + \lambda'\mu((\mathbf{m}',c'),D). \tag{4.1.3}$$

Proof. Note that inequality (4.1.3) follows directly from combining the other two statements.

We will first focus on equality (4.1.2). If $I_{\mathbf{m},c}$ and $I_{t\mathbf{m},c}$ are the maximal ideals of $R_{\mathbf{m},c}$ and $R_{t\mathbf{m},c}$, respectively, then $I^k_{t\mathbf{m},c} \cap R_{\mathbf{m},c} = I^{\lceil k/t \rceil}_{\mathbf{m},c}$ for every $k \in \mathbb{N}$, where we use the natural embedding $R_{\mathbf{m},c} \subset R_{t\mathbf{m},c}$. This directly implies the equality.

To prove inequality (4.1.1), we let $\mathcal{O}_{X,\tilde{c}}$ be the local ring at $\tilde{c}$ and define $R_{\mathbf{m},\tilde{c}}$ to be the ring obtained by adjoining all f_i^{1/m_i} with $m_i > 0$ to $\mathcal{O}_{X,\tilde{c}}$, where f_i is a local equation of D_i. (This is similar to the ring in Definition 4.1.9, but we only localize at $\tilde{c}$ rather than at c.) In order to compare the different rings $R_{\mathbf{m},\tilde{c}}, R_{\mathbf{m}',\tilde{c}}, R_{\mathbf{m}+\mathbf{m}',\tilde{c}}$, we embed them into the larger ring $R_{(\mathbf{m}+\mathbf{m}')\mathbf{m}\mathbf{m}',\tilde{c}}$, (where $(\mathbf{m}+\mathbf{m}')\mathbf{m}\mathbf{m}'$ is defined by coordinate-wise multiplication and addition). These inclusions fit into the commutative diagram

$$\begin{tikzcd}
 & R_{\mathbf{m},\tilde{c}} \arrow[dr, hook] & \\
\mathcal{O}_{X,\tilde{c}} \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & R_{\mathbf{m}+\mathbf{m}',\tilde{c}} \arrow[r, hook] & R_{(\mathbf{m}+\mathbf{m}')\mathbf{m}\mathbf{m}',\tilde{c}}. \\
 & R_{\mathbf{m}',\tilde{c}} \arrow[ur, hook] &
\end{tikzcd}$$

For $\tilde{\mathbf{m}} = \mathbf{m}, \mathbf{m}', \mathbf{m}+\mathbf{m}'$, we let $I_{\tilde{\mathbf{m}}}$ be the ideal of $R_{(\mathbf{m}+\mathbf{m}')\mathbf{m}\mathbf{m}',\tilde{c}}$ generated by

$$\{f_i^{1/\tilde{m}_i} \mid i \in \{1,\dots,n\}, \tilde{m}_i > 0\}.$$

Note that for all $\tilde{\mu} \in \mathbb{N}$, the ideal $I_{\tilde{\mathbf{m}}}^{\tilde{\mu}}$ is generated by the elements $\prod_{\substack{i=1\\ \tilde{m}_i>0}}^{n} f_i^{a_i/\tilde{m}_i}$ for $\mathbf{a} \in \mathbb{N}^n$ satisfying $\sum_{\substack{i=1\\ \tilde{m}_i>0}}^{n} a_i \geq \tilde{\mu}$. For ease of notation, let us assume that $m_i, m_i' > 0$ for all $i \in \{1, \dots, n\}$. For $\mu, \mu' \in \mathbb{N}$, the intersection $I_{\mathbf{m}}^{\mu} \cap I_{\mathbf{m}'}^{\mu'}$ is generated by elements x which can both be written as $x = \prod_{i=1}^{n} f_i^{a_i/m_i}$ with $\sum_{i=1}^{n} a_i \geq \mu$ and as $x = \prod_{i=1}^{n} f_i^{a_i'/m_i'}$ with $\sum_{i=1}^{n} a_i' \geq \mu'$, where $a_1, \dots, a_n, a_1', \dots, a_n' \in \mathbb{N}$.

For all $i \in \{1, \dots, n\}$, let v_i be the valuation on the fraction field $K(X)$ corresponding to the divisor D_i. Then $v_i(f_i) = 1$ and $v_i(f_j) = 0$ for all $j \neq i$ as D_i and D_j are distinct prime divisors on X. Up to scaling, this valuation extends to a discrete valuation $\tilde{v}_i$ on the fraction field of $R_{(\mathbf{m}+\mathbf{m}')\mathbf{m}\mathbf{m}',\tilde{c}}$ satisfying $\tilde{v}_i(f_i^{1/(m_i m_i'(m_i+m_i'))}) = 1$ and $\tilde{v}_i(f_j) = 0$ for all $j \neq i$. In particular, we see that the valuation of x with respect to $\tilde{v}_i$ is both $a_i m_i'(m_i + m_i')$ and $a_i' m_i(m_i + m_i')$ so $\frac{a_i}{m_i} = \frac{a_i'}{m_i'}$.

From this equality we obtain

$$x = \prod_{i=1}^{n} f_i^{(a_i+a_i')/(m_i+m_i')},$$

which implies that $x \in I_{\mathbf{m}+\mathbf{m}'}^{\mu+\mu'}$. Thus we conclude $I_{\mathbf{m}}^{\mu} \cap I_{\mathbf{m}'}^{\mu'} \subset I_{\mathbf{m}+\mathbf{m}'}^{\mu+\mu'}$, and by taking $\mu = \mu((\mathbf{m}, c), D)$ and $\mu' = \mu((\mathbf{m}', c'), D)$, we obtain inequality (4.1.1). $\square$

4.2 Geometry of the Picard group of a pair

In this section we introduce the Picard group and the effective cone of a pair.

Definition 4.2.1. We say that a divisor on (X, M) is *principal* if it is the image of a principal divisor on X under the homomorphism $\mathrm{pr}_M^*\colon \mathrm{Div}(X) \to \mathrm{Div}(X, M)$. We say that two divisors D, D' on (X, M) are *linearly equivalent* if $D - D'$ is a principal divisor.

Definition 4.2.2. We define the *Picard group* of (X, M) as

$$\mathrm{Pic}(X, M) = \mathrm{Div}(X, M)/\{\text{principal divisors}\}.$$

By the definition, the pullback $\mathrm{pr}_M^*\colon \mathrm{Div}(X) \to \mathrm{Div}(X, M)$ induces a homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X, M),$$

which we will also often denote by pr_M^*, by abuse of notation. We will denote the induced homomorphisms on $\mathbb{Q}$-divisors $\mathrm{Div}(X)_{\mathbb{Q}} \to \mathrm{Div}(X, M)_{\mathbb{Q}}$ and on $\mathbb{Q}$-divisor classes $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X, M)_{\mathbb{Q}}$ by pr_M^* as well. For a ($\mathbb{Q}$-)divisor D, we will denote the corresponding ($\mathbb{Q}$-)divisor class by $[D]$.

Definition 4.2.3. We define two $\mathbb{Q}$-divisors D, D' on a pair (X, M) to be $\mathbb{Q}$-*linearly equivalent* if the image of $D - D'$ in $\mathrm{Pic}(X, M)_{\mathbb{Q}}$ is 0.

Example 4.2.4. If (X, M) is a smooth pair corresponding to Darmon points with associated root stack $(X, \sqrt[\mathbf{m}]{D})$ as in Section 3.5, then [Cad07, Corollary 3.12] implies that the map $\mathrm{Pic}(X, M) \to \mathrm{Pic}(X, \sqrt[\mathbf{m}]{D})$, given by sending $\tilde{D}_{m_i\mathbf{e}}$ to the Cartier divisor

$\frac{1}{m_i}D_i$ for all $i \in \{1, \ldots, n\}$ with $m_i < \infty$, is an isomorphism. Furthermore, this isomorphism is compatible with the pullback homomorphisms of $\operatorname{Pic}(X) \to \operatorname{Pic}(X, M)$ and $\operatorname{Pic}(X) \to \operatorname{Pic}(X, \sqrt[\mathbf{m}]{D})$.

By taking all multiplicities $m_1, \ldots, m_n$ to be infinite, we obtain the following special case of the previous example.

Example 4.2.5. Let X be a smooth variety and let $U \subset X$ be an open subvariety such that $X \setminus U$ is a strict normal crossings divisor. If (X, M) is the smooth pair corresponding to integral points on U, then $\operatorname{Div}(X, M) = \operatorname{Div}(U)$ so $\operatorname{Pic}(X, M) = \operatorname{Pic}(U)$.

By construction of the Picard group of a pair, we have a natural surjection $\operatorname{Pic}(X, M) \to \operatorname{Pic}(U)$ by restricting a divisor on (X, M) to U. More generally, we can define restriction maps between the Picard groups of pairs.

Definition 4.2.6. Let (X, M) and (X, M') be two smooth pairs for the same choice of divisors $D_1, \ldots, D_n$, such that $\Gamma_{M'} \subset \Gamma_M$. Then we define the *restriction (of divisors) to* (X, M') to be the homomorphism

$$\operatorname{Div}(X, M) \to \operatorname{Div}(X, M')$$

which sends $\operatorname{Div}(X, M') \subset \operatorname{Div}(X, M)$ to itself and sends $D_{(\mathbf{m},c)}$ to 0 if $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$ but $\mathbf{m} \notin \Gamma_{M'}$. This homomorphism induces a homomorphism

$$\operatorname{Pic}(X, M) \to \operatorname{Pic}(X, M'),$$

which we will also refer to as the *restriction (of divisor classes) to* (X, M').

Note that these restrictions are always surjective homomorphisms. The cokernel of the homomorphism $\operatorname{pr}_M^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X, M)$ is generated by $[\tilde{D}_{\mathbf{m},c}]$ for $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$, and is thus finitely generated. Consequently, if $\operatorname{Pic}(X)$ is finitely generated, then $\operatorname{Pic}(X, M)$ will be as well, as it requires at most $\#\Gamma_{M,\mathcal{C}}$ additional generators.

Example 4.2.7. If (X, M) is a smooth pair for the Darmon points on the Campana pair $\left(X, \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) D_i\right)$, then $\operatorname{Div}(X, M)$ is naturally identified with $\operatorname{Div}(U) \oplus \mathbb{Z}\left(\frac{1}{m_1} D_1\right) \oplus \cdots \oplus \mathbb{Z}\left(\frac{1}{m_n} D_n\right)$, and the homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X, M)$ is injective with cokernel

$$\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}.$$

Example 4.2.8. In the previous example, if we take $X = \mathbb{P}_K^{n-1}$, and we let $D_1, \ldots, D_n$ be the coordinate hyperplanes, then

$$\operatorname{Pic}(X, M) \cong \mathbb{Z}^n / \left\{ (a_1, \ldots, a_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n \frac{a_i}{m_i} = 0 \right\}.$$

In particular, if $\gcd(m_i, m_j) = 1$ for all distinct $i, j \in \{1, \ldots, n\}$, then $\operatorname{Pic}(X, M) \cong \mathbb{Z}$. On the other hand, if $m = m_1 = \cdots = m_n$, then $\operatorname{Pic}(X, M) \cong \mathbb{Z} \times (\mathbb{Z}/m\mathbb{Z})^{n-1}$. This shows that $\operatorname{Pic}(X, M)$ can contain nontrivial torsion even when $\operatorname{Pic}(X)$ is torsion-free.

The previous example showed that the torsion of $\operatorname{Pic}(X, M)$ is slightly subtle, and depends on M, rather than only on X. However, for a proper pair, the rank of its Picard group is very simple to describe.

Proposition 4.2.9. *If (X, M) is a proper smooth pair over a field K, then pr_M^* is injective on $\operatorname{Div}(X)$ and on $\operatorname{Pic}(X)$. Moreover*

$$\operatorname{rank} \operatorname{Pic}(X, M) = \operatorname{rank} \operatorname{Pic}(X) + \#\Gamma_{M,\mathcal{C}} - n.$$

Proof. Since (X, M) is proper, for each $i = 1, \dots, n$ there exists an element $\mathbf{m}_i := m_i \mathbf{e}_i \in \Gamma_M$ for some integer $m_i > 0$. In particular, the coefficient of $\tilde{D}_{\mathbf{m}_i,c}$ in $\operatorname{pr}_M^* D_i$ is m_i for $c = D_i$, while for all $j \in \{1, \dots, n\}$ different from i, the coefficient of $\tilde{D}_{\mathbf{m}_i,c}$ in $\operatorname{pr}_M^* D_j$ is zero, as D_i is not contained in D_j. Similarly, for all $D \in \operatorname{Div}(U)$, the the coefficient of $\tilde{D}_{\mathbf{m}_i}$ in $\operatorname{pr}_M^* D$ is also zero. Since, furthermore, the restriction of pr_M^* to $\operatorname{Div}(U)$ is injective, pr_M^* is indeed injective. This directly implies that the homomorphism on the Picard groups is also injective.

If we let $(X, M') \subset (X, M)$ be the pair with $\mathfrak{M}' = \{(0, \dots, 0), \mathbf{m}_1, \dots, \mathbf{m}_n\}$, then we can consider the restriction $\operatorname{Pic}(X, M) \to \operatorname{Pic}(X, M')$. The kernel of this restriction is the free abelian group generated by the divisor classes $[\tilde{D}_{\mathbf{m},c}]$ for $(\mathbf{m}, c) \in \Gamma_{M,\mathcal{C}}$ with $\mathbf{m} \notin \Gamma_{M'}$. In particular, it is a free abelian group of rank $\#\Gamma_{M,\mathcal{C}} - n$. By Example 4.2.7, the cokernel of $\operatorname{pr}_{M'}^* \colon \operatorname{Pic}(X) \to \operatorname{Pic}(X, M')$ is finite. As (X, M') is proper, this implies $\operatorname{rank} \operatorname{Pic}(X, M) = \operatorname{rank} \operatorname{Pic}(X)$, so the rank of $\operatorname{Pic}(X, M)$ is $\operatorname{rank} \operatorname{Pic}(X) + \#\Gamma_{M,\mathcal{C}} - n$. □

Now we will define the canonical divisor class of a pair.

Definition 4.2.10. For a smooth pair (X, M) over a field K of characteristic 0, we define the *ramification divisor* of (X, M) to be the effective divisor

$$R := \sum_{(\mathbf{m},c) \in \Gamma_{M,\mathcal{C}}} \left(-1 + \sum_{i=1}^{n} m_i \right) \tilde{D}_{\mathbf{m},c}.$$

The *canonical (divisor) class* $K_{(X,M)} \in \operatorname{Pic}(X, M)$ of a smooth pair (X, M) is

$$\begin{aligned} K_{(X,M)} &:= \operatorname{pr}_M^* K_X + R \\ &= \operatorname{pr}_M^* \left(K_X + \sum_{i=1}^{n} [D_i] \right) - \sum_{(\mathbf{m},c) \in \Gamma_{M,\mathcal{C}}} [\tilde{D}_{\mathbf{m},c}]. \end{aligned}$$

Thus the canonical class is defined in such a way that it satisfies an analogue of the Hurwitz formula for morphisms of curves (see e.g. [Har77, Chapter IV, Proposition 2.3]). If (X, M) is a smooth pair corresponding to Darmon points on a Campana pair $(X, D_{\mathbf{m}})$, then the canonical divisor of (X, M) agrees with the canonical divisor of the root stack $(X, \sqrt[\mathbf{m}]{D})$, see for example [VZ22, Proposition 5.5.6] for the case when X is a curve.

Assumption 4.2.11. For the rest of the thesis we assume that X is a rationally connected proper variety over a field of characteristic 0.

There are two reasons for this assumption. One reason is that Conjecture 1.2.2 only applies to rationally connected varieties. Furthermore, if X is rationally connected then the Albanese variety of X is trivial, and thus its dual $\operatorname{Pic}^0(X)$ is also trivial. Hence, $\operatorname{Pic}(X, M)$ is a finitely generated abelian group for every smooth pair (X, M). We will now introduce the effective cone.

Definition 4.2.12. The *effective cone of a smooth pair* (X, M) is the cone

$$\operatorname{Eff}^1(X, M) \subset \operatorname{Pic}(X, M)_{\mathbb{R}}$$

generated by effective divisors on (X, M). Its topological closure is the *pseudo-effective cone of* (X, M)

$$\overline{\operatorname{Eff}}^1(X, M) \subset \operatorname{Pic}(X, M)_{\mathbb{R}}.$$

We also similarly write $\operatorname{Eff}^1(X)$ and $\overline{\operatorname{Eff}}^1(X)$ for the effective and pseudo-effective cones of X.

The effective cone of a proper pair is strongly convex, as the next proposition shows.

Proposition 4.2.13. *Let* (X, M) *be a smooth proper pair. Then* $\operatorname{Eff}^1(X, M)$ *is strictly convex, i.e.*

$$\operatorname{Eff}^1(X, M) \cap -\operatorname{Eff}^1(X, M) = \{\mathbf{0}\}.$$

Proof. We will argue by contradiction. Suppose that there exists an element $E \in \operatorname{Eff}^1(X, M)$ satisfying $-E \in \operatorname{Eff}^1(X, M)$. Since $\operatorname{Eff}^1(X, M)$ is generated by effective divisors, $\operatorname{Eff}^1(X, M) \cap \mathbb{Q}$ is dense in $\operatorname{Eff}^1(X, M)$. This implies that there exists an integer m and nonzero effective divisors D_1, D_2 on (X, M) such that $mE - [D_1], -mE - [D_2] \in \operatorname{Eff}^1(X, M)$. The sum $D := D_1 + D_2$ is a nonzero effective divisor such that $-[D] \in \operatorname{Eff}^1(X, M)$. This implies that $-D$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor D'. It follows that $D + D'$ is an effective $\mathbb{Q}$-divisor which is $\mathbb{Q}$-linearly equivalent to 0. Thus there exists a positive integer m such that $m(D+D')$ is linearly equivalent to an effective divisor and $m(D + D') = \operatorname{pr}_M^* \operatorname{div}(f)$ for some rational function on X. The function f cannot have any poles on U, and since (X, M) is proper, there exist positive integers $m_1, \dots, m_n$ such that $m_1\mathbf{e}_1, \dots, m_n\mathbf{e}_n \in \Gamma_M$. This implies that f cannot have any poles at the divisors $D_1, \dots, D_n$ either, which implies that f is a regular function on X and thus f is constant, as X is proper. We conclude that $D + D' = 0$. This contradicts the fact that D is nonzero, so E cannot exist. □

Using the effective cone, we will now define the Fujita invariant and the b-invariant of a pair.

Definition 4.2.14. Let (X, M) be a smooth pair over a field K of characteristic 0. Let L be a nef and big $\mathbb{Q}$-divisor class on X. We define the *Fujita invariant* of (X, M) with respect to L to be

$$a((X, M), L) = \inf\{t \in \mathbb{R} \mid t \operatorname{pr}_M^* L + K_{(X,M)} \in \overline{\operatorname{Eff}}^1(X, M)\}.$$

We call $a((X, M), L) \operatorname{pr}_M^* L + K_{(X,M)} \in \overline{\operatorname{Eff}}^1(X, M)$ the *adjoint divisor class of* L *with respect to* (X, M). We define the b-*invariant* $b(K, (X, M), L)$ to be the codimension of the minimal supported face of $\overline{\mathrm{Eff}}^1(X,M)$ which contains the adjoint divisor class of L with respect to (X,M).

Note that the Fujita invariant is strictly positive if and only if $K_{(X,M)}$ is not pseudoeffective.

That there need not exist a nef and big $\mathbb{Q}$-divisor class L such that $\mathrm{pr}_M^* L = -K_{(X,M)}$, so the b-invariant may be smaller than the rank of the Picard group of (X,M) for all choices of L, as the next example illustrates.

Example 4.2.15. Let (X,M) be the smooth pair over K corresponding to the Campana points on $\left(X, \sum_{i=1}^n \left(1-\frac{1}{m_i}\right) D_i\right)$ for a rationally connected variety X. Then Proposition 4.2.9 implies that

$$\operatorname{rank}\operatorname{Pic}(X,M) = \operatorname{rank}\operatorname{Pic}(X) + \sum_{i=1}^{n}(m_i-1),$$

where we use the description of Γ_M from Example 4.1.6 and the fact that the set of components $\mathcal{C}_{m\mathbf{e}_i}$ is just $\{D_i\}$ for any positive integer m and $i \in \{1,\dots,n\}$. However, for any divisor $L \in \mathrm{Div}(X)$, the coefficient of $\tilde{D}_{m\mathbf{e}_i}$ in L for $m_i \leq m \leq 2m_i - 1$ is simply $a_i m$, where a_i is the coefficient of D_i in L. This follows from the equality (4.1.2) combined with the fact that $\mu(m\mathbf{e}_i, D) = 0$ for all prime divisors $D \neq D_i$ on X. Therefore, if the coefficient of $\tilde{D}_{m_i\mathbf{e}_i}$ in $\mathrm{pr}_M^* L + R = \mathrm{pr}_M^* L + \sum_{i=1}^n \sum_{m=m_i}^{2m_i-1}(m-1)\tilde{D}_{m\mathbf{e}_i}$ is nonnegative for some $i \in \{1,\dots,n\}$, then the coefficient of $\tilde{D}_{m\mathbf{e}_i}$ in $\mathrm{pr}_M^* L + R$ is positive for integers m with $m_i < m \leq 2m_i - 1$. Consequently, we have $b(K,(X,M),L) \leq \operatorname{rank}\operatorname{Pic}(X)$ for all big and nef divisors L.

In fact, for computing the Fujita invariant and the b-invariant, we only need to consider small generators $\mathbf{m} \in \Gamma_M$, which often considerably simplifies computations.

Lemma 4.2.16. *Let (X,M) be a smooth pair over a field K of characteristic 0 and let L be a big and nef $\mathbb{Q}$-divisor on X. Let P be the polyhedron given as the convex hull of the set*

$$\{\mathbf{m} + \mathbf{x} \in \mathbb{R}^n \mid \mathbf{m} \in \Gamma_M, \mathbf{x} \in \mathbb{R}^n_{\geq 0}\}$$

and let ∂P and $V(P)$ be its boundary and its set of vertices respectively. Define pairs $(X,M'') \subset (X,M') \subset (X,M)$ by setting $\mathfrak{M}' = \Gamma_M \cap \partial P$ and $\mathfrak{M}'' = \Gamma_M \cap V(P)$. Then

$$a((X,M),L) = a((X,M'),L) = a((X,M''),L)$$

and

$$b(K,(X,M),L) = b(K,(X,M'),L).$$

Proof. If $\Gamma_{M,\mathcal{C}} = \emptyset$, then P is empty and $(X,M) = (X,M') = (X,M'')$ is the pair with $\mathfrak{M} = \{(0,\dots,0)\}$, so the lemma is trivially satisfied. Now assume $\Gamma_{M,\mathcal{C}} \neq \emptyset$ and let $(\mathbf{m},c) \in \Gamma_{M,\mathcal{C}}$ be an element such that $\mathbf{m}$ is not on the boundary of the polyhedron. Then there exist $(\mathbf{m}_1,c_1),\dots,(\mathbf{m}_T,c_T) \in \Gamma_{M',\mathcal{C}}$ such that $\mathbf{m} = \sum_{t=1}^T \lambda_t \mathbf{m}_t$ for some real numbers $\lambda_1,\dots,\lambda_T > 0$ satisfying $\sum_{t=1}^T \lambda_t > 1$ and such that c is a component of $\bigcap_{t=1}^T c_t$. If $L' \in \mathrm{Div}(X)_{\mathbb{Q}}$ is a $\mathbb{Q}$-divisor, such that for all $t \in \{1,\dots,T\}$ the coefficient of $\tilde{D}_{\mathbf{m}_t,c_t}$ in $\mathrm{pr}_M^* L' \in \mathrm{Div}(X,M)$ is at least 1, then Lemma 4.1.19 implies that the coefficient of $\tilde{D}_{\mathbf{m},c}$ in $\mathrm{pr}_M^* L' \in \mathrm{Div}(X,M)$ is at least $\sum_{t=1}^T \lambda_t > 1$. This implies that for any $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ and $a \in \mathbb{Q}$ such that

$$a\,\mathrm{pr}_M^* L + K_{(X,M)} = \mathrm{pr}_M^* \left(L + K_X + \sum_{i=1}^n [D_i] \right) - \sum_{(\mathbf{m},c)\in\Gamma_{M,\mathcal{C}}} [\tilde{D}_{\mathbf{m},c}] \in \mathrm{Pic}(X,M)_{\mathbb{Q}}$$

is an effective $\mathbb{Q}$-divisor class, $a\,\mathrm{pr}_M^* L + K_{(X,M)}$ is represented by an effective $\mathbb{Q}$-divisor such that the coefficient of $\tilde{D}_{\mathbf{m},c}$ is at least $-1 + \sum_{t=1}^T \lambda_t > 0$. This implies that the minimal face of $\overline{\mathrm{Eff}}^1(X,M)$ containing the adjoint divisor class of an big and nef $\mathbb{Q}$-divisor class L contains $\tilde{D}_{\mathbf{m},c}$. Thus we see that $a((X,M),L) = a((X,M'),L)$ and $b(K,(X,M),L) = b(K,(X,M'),L)$, as desired. The proof of the equality $a((X,M),L) = a((X,M''),L)$ is entirely analogous. $\square$

More generally, the Fujita invariant and the b-invariant are smaller on smaller pairs.

Proposition 4.2.17. *Let $(X,M) \subset (X,M')$ be smooth pairs over a field K of characteristic 0 and let L be a big and nef $\mathbb{Q}$-divisor on X. Then we have*

$$(a((X,M),L), b(K,(X,M),L)) \leq (a((X,M'),L), b(K,(X,M'),L))$$

in the lexicographic ordering.

Proof. The proof is essentially the same argument as in Lemma 4.2.16 by using Lemma 4.1.19. $\square$

Remark 4.2.18. By Lemma 4.2.16, the Fujita invariant and the b-invariant only depend on the polyhedron generated by $\mathfrak{M}$, rather than $\mathfrak{M}$ itself. Therefore the Conjecture 1.2.2 satisfies a form of purity: the order of growth of the counting function only depends on the smallest elements in $\mathfrak{M}$.

Example 4.2.19. As simple example for the previous remark, Conjecture 1.2.2 implies that the $\mathbb{Q}$-rational points in projective space with coordinates both squarefree and pairwise coprime have a positive density in the full set of rational points, when the points are ordered by their Weil height.

There is a natural generalization of rigid divisors to pairs.

Definition 4.2.20. Let $D \in \mathrm{Div}(X,M)_{\mathbb{Q}}$ be a $\mathbb{Q}$-divisor on a smooth pair (X,M) with X rationally connected. We say that D is *rigid* if D is effective and it is the only effective $\mathbb{Q}$-divisor in its $\mathbb{Q}$-linear equivalence class. For a big and nef $\mathbb{Q}$-divisor L on X, we say that L is *adjoint rigid* with respect to (X,M) if $a((X,M),L) > 0$ and the adjoint divisor class $a((X,M),L)\,\mathrm{pr}_M^* L + K_{(X,M)}$ is represented by a rigid $\mathbb{Q}$-divisor.

In Chapter 5, we will consider non-proper pairs to which our counting methods for $\mathcal{M}$-points apply, by imposing a condition on the pair depending on the $\mathbb{Q}$-divisor defining the height. Such pairs will play an instrumental role in proving Theorem 1.2.7 for proper pairs.

Definition 4.2.21. Let X be a smooth proper variety and let $L \in \operatorname{Pic}(X)_{\mathbb{Q}}$ be a big and nef $\mathbb{Q}$-divisor class. A smooth pair (X, M) is *quasi-proper with respect to L* if there exists a smooth proper pair $(X, \tilde{M})$ containing (X, M) such that $a((X, M), L) = a((X, \tilde{M}), L)$.

Proposition 4.2.22. *Let (X, M) be a smooth pair which is quasi-proper with respect to some big and nef $\mathbb{Q}$-divisor class. Then a proper pair $(X, \tilde{M})$ as in Definition 4.2.21 can be found by adjoining the elements $m\mathbf{e}_1, \ldots, m\mathbf{e}_n$ to $\mathfrak{M}$ for any sufficiently large integer m.*

Proof. For any proper pair $(X, \overline{M})$ satisfying $a((X, M), L) = a((X, \overline{M}), L)$, there are positive integers $m_1, \ldots, m_n$ such that $m_1\mathbf{e}_1, \ldots, m_n\mathbf{e}_n \in \overline{\mathfrak{M}}$, and Proposition 4.2.17 implies that the pair (X, M') given by $\mathfrak{M}' = \{m_1\mathbf{e}_1, \ldots, m_n\mathbf{e}_n\} \cup \mathfrak{M}$ has the same Fujita invariant as $(X, \overline{M})$ and (X, M). Now let $m \geq m_1, \ldots, m_n$ be an integer and let $(X, \tilde{M})$ be the pair given by $\tilde{\mathfrak{M}} = \{m\mathbf{e}_1, \ldots, m\mathbf{e}_n\} \cup \mathfrak{M}$. The pair (X, M'') given by $\mathfrak{M}'' = \mathfrak{M}' \cup \tilde{\mathfrak{M}}$ has the same Fujita invariant as (X, M') by Lemma 4.2.16, and Proposition 4.2.17 implies

$$a((X, \tilde{M}), L) \leq a((X, M''), L) = a((X, M), L).$$

Now since $(X, M) \subset (X, \tilde{M})$, we must have $a((X, \tilde{M}), L) = a((X, M), L)$ by Proposition 4.2.17. $\square$

For a pair $(X, \tilde{M})$ as in Proposition 4.2.22, there is an inclusion $\Gamma_M \subset \Gamma_{\tilde{M}}$ giving a restriction homomorphism $\operatorname{Pic}(X, \tilde{M}) \to \operatorname{Pic}(X, M)$ as in Definition 4.2.6. In this setting, we can view the condition $a((X, M), L) = a((X, \tilde{M}), L)$ as the statement that the adjoint divisor of L with respect to (X, M) is the restriction of the adjoint divisor of L with respect to $(X, \tilde{M})$.

Example 4.2.23. Let X be a rationally connected smooth proper variety such that $-K_X$ is big, and let D be a strict normal crossings divisor on X which is rigid. Then the pair (X, M) corresponding to the integral points on the open $U = X \setminus D$ is quasi-proper for any any big and nef $\mathbb{Q}$-divisor class $L = -K_X + aD$ with $a > -1$. We can take the proper pair $(X, \tilde{M})$ to be the pair corresponding to the Darmon points on $\left(X, \sum_{i=1}^n \left(1 - \frac{1}{m}\right) D_i\right)$, where $D_1, \ldots, D_n$ are the irreducible components of D and m is a positive integer such that $-1 + \frac{1}{m} \leq a$. In particular, if $a \geq 0$, then we can take $(X, \tilde{M})$ to be the trivial pair. This follows from the rigidity of D combined with the simple calculation

$$\operatorname{pr}_{\tilde{M}}^* L + K_{(X, \tilde{M})} = \sum_{i=1}^n (am + (m-1)) \tilde{D}_{m\mathbf{e}_i},$$

which corresponds to $\sum_{i=1}^n \left(a + 1 - \frac{1}{m}\right) D_i$ under the identification in Example 4.2.4. This implies $a((X, M), L) = a((X, \tilde{M}), L) = 1$, as desired.

Example 4.2.24. As a special case of the previous example, we can take $X = \operatorname{Bl}_{(0:0:1)} \mathbb{P}^2$, let D_1 be the exceptional divisor and take $\mathfrak{M} = \{0\}$. Then for any $a \in (-1, 1] \cap \mathbb{Q}$, the $\mathbb{Q}$-divisor class $L = K_X + aD$ is big and nef, and (X, M) is quasi-proper with respect to L.

4.3 Rationally connected pairs

In modern times, Manin's Conjecture is often formulated for rationally connected varieties, see for example [LST22, Conjecture 1.2]. The notion of rationally connected varieties (see e.g. [KMM92, Definition-Remark 2.2]) has a natural extension to smooth proper pairs.

Definition 4.3.1. A smooth proper pair (X, M) over a field K is *rationally connected* if there exists a nonempty open subvariety $V \subset X$ such that for each algebraically closed field L/K and every two points $p_1, p_2 \in V(L)$, there exists a rational curve $C \subset X_L$ containing both points such that C is the image of a morphism $f \in (X, M_{\mathrm{mon}})(\mathbb{P}^1_L)$, where $(X, M_{\mathrm{mon}})(\mathbb{P}^1_L)$ is as in Definition 2.1.17.

In other words, a pair (X, M) is rationally connected if for any two general points on X there is a projective rational curve passing through them respecting the conditions imposed by $M_{\mathbf{m}}$. In particular, if (X, M) is rationally connected, then X is rationally connected as well. The projectivity of the curve is crucial here, as any curve C on X not contained in the divisors $D_1, \dots, D_n$ has a nonempty open subset $C' \subset C$ avoiding these divisors.

One reason to consider such pairs in Conjecture 1.2.2 is that they have a good reason for having plenty of $\mathcal{M}$-points after an extension of the ground field. This is because the images of rational points under $f \in (X, M_{\mathrm{mon}})(\mathbb{P}^1_K)$ are $\mathcal{M}$-points over $\mathcal{O}_S$ for some finite set of places $S \subset \Omega_K$, as the next proposition shows.

Proposition 4.3.2. *Let (X, M) be a pair over a number field K and let $f \in (X, M_{\mathrm{mon}})(\mathbb{P}^1_K)$. Then for every finite set of places $S \subset \Omega_K$ and every $\mathcal{O}_S$-integral model $(\mathcal{X}, \mathcal{M})$, there exists a finite set of places $S' \supset S$ such that f extends to $\tilde{f} \in (\mathcal{X}, \mathcal{M}_{\mathrm{mon}})(\mathbb{P}^1_{\mathcal{O}_{S'}})$. Furthermore, for any $P \in \mathbb{P}^1_K(K)$,*

$$f(P) \in (\mathcal{X}, \mathcal{M}_{\mathrm{mon}})(\mathcal{O}_{S'})$$

if $f(P)$ lies in U.

Proof. By spreading out [Poo17, Theorem 3.2.1], we can find a set of places S' containing S such that $f \colon \mathbb{P}^1_K \to X$ lifts to a morphism $\tilde{f} \colon \mathbb{P}^1_{\mathcal{O}_{S'}} \to \mathcal{X}$. The pullbacks $\tilde{f}^*\mathcal{D}_1, \dots, \tilde{f}^*\mathcal{D}_n$ are effective divisors on $\mathbb{P}^1_{\mathcal{O}_{S'}}$. By further enlarging S' if necessary, we can ensure that these divisors have no components supported above a prime $\mathfrak{p}$ in $\mathcal{O}_{S'}$, which ensures that $\tilde{f} \in (\mathcal{X}, \mathcal{M}_{\mathrm{mon}})(\mathbb{P}^1_{\mathcal{O}_{S'}})$. Let $P \in \mathbb{P}^1_K(K)$. By the valuative criterion of properness, P corresponds to an unique integral point $\tilde{P} \colon \operatorname{Spec} \mathcal{O}_{S'} \to \mathbb{P}^1$. If $\mathfrak{p}$ is a prime of $\mathcal{O}_{S'}$, and we denote the coefficient of $\mathfrak{p}$ in $(\tilde{f} \circ \tilde{P})^*\mathcal{D}_i = \tilde{P}^*(\tilde{f}^*\mathcal{D}_i)$ by m_i for all $i = 1, \dots, n$, then $(m_1, \dots, m_n) \in \mathfrak{M}_{\mathrm{mon}}$ since $\mathfrak{M}_{\mathrm{mon}}$ is a monoid. $\square$

4.4 Quasi-Campana points and the log-canonical divisor

In this section we specialize the Conjecture 1.2.2 to Campana points, weak Campana points and Darmon points as defined in Definition 2.1.19. In particular, we will clarify the relation of Conjecture 1.2.2 with the conjecture [PSTVA21, Conjecture 1.1] on Campana points of bounded height. In order to uniformly discuss these different kinds of points, we introduce the notion of a quasi-Campana pair.

Definition 4.4.1. Let $m_1, \dots, m_n \in \mathbb{N} \cup \{\infty\}$. A pair (X, M) is *quasi-Campana* for the Campana pair $(X, D_{\mathbf{m}})$, where $D_{\mathbf{m}} = \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) D_i$, if the following conditions are satisfied:

1. for all $i \in \{1, \dots, n\}$ with $m_i = \infty$ we have $w_i = 0$ for all $\mathbf{w} \in \mathfrak{M}$,
2. $m_i \mathbf{e}_i \in \mathfrak{M}$ for all $i \in \{1, \dots, n\}$ with $m_i < \infty$, where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{Z}^n$,
3. for all $\mathbf{w} \in \mathfrak{M} \setminus \{0, \dots, 0\}$, $\sum_{i=1}^n \frac{w_i}{m_i} \geq 1$.

Examples of quasi-Campana pairs are given by the pairs for (weak) Campana points and Darmon points. In the theory of Campana points, the log-canonical class $K_X + D_{\mathbf{m}}$ plays a crucial role. We give an intrinsic definition of the log-canonical class on a pair (X, M) as the "best approximation from below" of the canonical class $K_{(X,M)}$ by a $\mathbb{Q}$-divisor class on X.

Definition 4.4.2. Let (X, M) be a smooth proper pair over a field K such that X is rationally connected. A $\mathbb{Q}$-divisor class D on X is called the *log-canonical class* for (X, M) if $K_{(X,M)} - \mathrm{pr}_M^* D \in \mathrm{Eff}^1(X, M)$ and for any $\mathbb{Q}$-divisor class D' satisfying the same property, we have $D - D' \in \mathrm{Eff}^1(X)$. We will write $K_{(X,M),\log}$ for the log-canonical class.

Note that since $\mathrm{Eff}^1(X, M)$ is strictly convex by Proposition 4.2.13, any two divisors whose classes are log-canonical are $\mathbb{Q}$-linearly equivalent, so the above definition makes sense.

For the pair corresponding to Campana points, the log-canonical divisor is simply $K_X + D_{\mathbf{m}}$.

Proposition 4.4.3. *Let (X, M) be a smooth proper quasi-Campana pair for the Campana pair $(X, D_{\mathbf{m}})$. Then $K_{(X,M),\log} = K_X + D_{\mathbf{m}}$ is the log-canonical divisor class for (X, M). Furthermore, if (X, M) is a pair corresponding to Darmon points, then $\mathrm{pr}_M^* K_{(X,M),\log} = K_{(X,M)}$.*

Proof. The definition of the log-canonical class $K_{(X,M)}$ immediately implies that $K_{(X,M)} - \mathrm{pr}_M^*(K_X + D_{\mathbf{m}})$ is an effective $\mathbb{Q}$-divisor class on (X, M). Additionally, if (X, M_{Darmon}) is the pair corresponding to Darmon points on $(X, D_{\mathbf{m}})$ then $\mathrm{pr}_{M_{\mathrm{Darmon}}}^*(K_X + D_{\mathbf{m}}) = K_{(X,M_{\mathrm{Darmon}})}$, so $K_X + D_{\mathbf{m}}$ is the log-canonical class for (X, M_{Darmon}). For other quasi-Campana pairs for the Campana pair $(X, D_{\mathbf{m}})$, the inclusion $\Gamma_{M_{\mathrm{Darmon}},\mathcal{C}} = \{m_1\mathbf{e}_1, \dots, m_n\mathbf{e}_n\} \subset \Gamma_{M,\mathcal{C}}$ induces a group homomorphism $\mathrm{Pic}(X, M) \to \mathrm{Pic}(X, M_{\mathrm{Darmon}})$ compatible with the pullback homomorphisms pr_M^* and $\mathrm{pr}_{M_{\mathrm{Darmon}}}^*$. This map sends effective divisor classes to effective divisor classes and $K_{(X,M)}$ to $K_{(X,M_{\mathrm{Darmon}})}$. Thus for any $\mathbb{Q}$-divisor class D on X such that $K_{(X,M)} - \mathrm{pr}_M^* D$ is effective, $K_{(X,M_{\mathrm{Darmon}})} - \mathrm{pr}_{M_{\mathrm{Darmon}}}^* D$ is effective as well. This implies $D_{\mathbf{m}} - D \in \mathrm{Eff}^1(X)$, as we saw that $K_X + D_{\mathbf{m}}$ is the log-canonical class for (X, M_{Darmon}). Therefore $K_X + D_{\mathbf{m}}$ is the log-canonical class for (X, M). $\square$

There are various other pairs for which the log-canonical class exists, as the next example shows.

Example 4.4.4. Let (X, M) be a smooth proper pair such that $\operatorname{Pic}(X)_{\mathbb{Q}} \cong \mathbb{Q}$ and such that $\operatorname{Eff}^1(X, M)$ is a rational polyhedral cone. Then there exists a log-canonical divisor on X for (X, M). This is because for any nonzero effective divisor D on X there is a largest $a \in \mathbb{Q}$ that satisfies $K_{(X,M)} - \operatorname{pr}_M^* a[D] \geq 0$, so $K_{(X,M),\log} = a[D]$ is the log-canonical class. Here the assumption on the effective cone ensures that aD is a $\mathbb{Q}$-divisor, rather than just an $\mathbb{R}$-divisor.

However, the log-canonical class need not exist if the pair is not quasi-Campana, as the next example shows.

Example 4.4.5. Let $X = \operatorname{Bl}_{(0:0:1)} \mathbb{P}^2$. Let $D_1 = D$ be the strict transform of a line passing through $(0 : 0 : 1) \in \mathbb{P}^2$ and let $D_2 = E$ be the exceptional divisor. Let $\mathfrak{M} \subset \mathbb{N}^2$ be the monoid generated by $(3, 0)$, $(0, 3)$ and $(1, 1)$. Then the effective cone of X is generated by D and E, and a $\mathbb{Q}$-divisor $D' = (-2 - a)D + (-1 - b)E = K_X + (1 - a)D + (1 - b)E$ satisfies $K_{(X,M)} - \operatorname{pr}_M^* D' \geq 0$ if and only if $3a \geq 1$, $3b \geq 1$ and $a + b \geq 1$. There is no solution (a, b) to this system of inequalities with a and b simultaneously minimal. Therefore there does not exist a log-canonical class on X for (X, M).

For determining the Fujita invariant of a divisor class, we can use the log-canonical class rather than the canonical class of the pair.

Proposition 4.4.6. *Let (X, M) be a smooth proper pair over a field K. Assume that that there exists a log-canonical class $K_{(X,M),\log}$ for (X, M). Let L be a big and nef $\mathbb{Q}$-divisor class on X. Then*

$$a((X, M), L) = \inf\{t \in \mathbb{R} \mid tL + K_{(X,M),\log} \in \overline{\operatorname{Eff}}^1(X)\}.$$

Proof. Since L is big, and thus an effective $\mathbb{Q}$-divisor class, the infimum of all $t \in \mathbb{R}$ such that $tL + K_{(X,M),\log}$ is effective is the same as the infimum of all $t \in \mathbb{R}$ such that $tL + K_{(X,M),\log}$ is pseudo-effective. For any $\mathbb{Q}$-divisor class D on X, we have $D + K_{(X,M),\log} \in \operatorname{Eff}^1(X)$ if and only if $\operatorname{pr}_M^* D + K_{(X,M)} \in \operatorname{Eff}^1(X, M)$, by the definition of the log-canonical class applied to $D' = -D$. By taking $D = tL$, we find the desired identity for $a((X, M), L)$. □

Similarly, we can use the log-canonical class to determine whether a divisor class is adjoint rigid.

Proposition 4.4.7. *Let (X, M) be a proper quasi-Campana pair and let L be a big and nef $\mathbb{Q}$-divisor on X. Then $a((X, M), L) \operatorname{pr}_M^* L + K_{(X,M)}$ is rigid if and only if $a((X, M), L)L + K_X + D_{\mathbf{m}}$ is rigid.*

Proof. Let $D' \in \operatorname{Div}(X)$ be a representative of the canonical divisor K_X, and set $D = \operatorname{pr}_M^* D' + \sum_{(\mathbf{m},c)\in\Gamma_{M,c}} (-1 + \sum_{i=1}^n m_i)\, \tilde{D}_{\mathbf{m},c}$ and $D_{\log} = D' + D_{\mathbf{m}}$. Then D, $D_{\log}$ represent the $\mathbb{Q}$-divisor classes $K_{(X,M)}$ and $K_X + D_{\mathbf{m}}$ on (X, M) and X, respectively. A direct calculation shows that $D - \operatorname{pr}_M^* D_{\log}$ is an effective divisor which vanishes on U and on the divisors corresponding to the elements $m_i\mathbf{e}_i \in \mathfrak{M}$. Thus if $E \in \operatorname{Div}(X)_{\mathbb{Q}}$

and $\tilde{D}$ is either a prime divisor on U or $\tilde{D} = \tilde{D}_{m_i\mathbf{e}_i}$ for $i \in \{1, \dots, n\}$, then the coefficient of $\tilde{D}$ in $\mathrm{pr}_M^*(D_{\log} + E)$ is equal to the coefficient of $\tilde{D}$ in $D + \mathrm{pr}_M^* E$. Since the coefficient of $\tilde{D}_{m_i\mathbf{e}_i}$ in $\mathrm{pr}_M^*(D_{\log} + E)$ is m_i times the coefficient of D_i in $D_{\log} + E$ for all $i \in \{1, \dots, n\}$, this implies that $D_{\log} + E$ is effective if and only if $D + \mathrm{pr}_M^* E$ is effective. Therefore, $D_{\log} + E$ is rigid if and only if $D + \mathrm{pr}_M^* E$ is rigid. □

However, in general the log-canonical class need not be adjoint rigid when it exists.

Example 4.4.8. Let $X = \mathbb{P}^2_K$ and let D_1 and D_2 be two distinct lines in $\mathbb{P}^2_K$. As in Example 4.4.5 we let $\mathfrak{M} \subset \mathbb{N}^2$ be the monoid generated by $(3,0)$, $(0,3)$ and $(1,1)$. The canonical divisor class $K_{(X,M)}$ of (X,M) is represented by the divisor

$$\begin{aligned}\tilde{D} &= -\mathrm{pr}_M^*(-2D_1 - D_2 + D_1 + D_2) - \tilde{D}_{(3,0)} - \tilde{D}_{(0,3)} - \tilde{D}_{(1,1)} \\ &= -4\tilde{D}_{(3,0)} - \tilde{D}_{(0,3)} - 2\tilde{D}_{(1,1)}.\end{aligned}$$

The Picard group of the pair, $\mathrm{Pic}(X,M) \cong \mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$, is generated by the divisors $\tilde{D}_{(3,0)}, \tilde{D}_{(0,3)}, \tilde{D}_{(1,1)}$, where $3\tilde{D}_{(3,0)}$ is linearly equivalent to $3\tilde{D}_{(0,3)}$. Both $E_1 = -\frac{4}{3}D_1 - \frac{2}{3}D_2$ and $E_2 = -\frac{5}{3}D_1 - \frac{1}{3}D_2$ represent the log-canonical class $K_{(X,M),\log}$. Since $\mathrm{pr}_M^* E_1 + \tilde{D}$ and $\mathrm{pr}_M^* E_2 + \tilde{D}$ are effective, E_1 and E_2 are not adjoint rigid for the pair (X,M).

Remark 4.4.9. For a quasi-Campana pair, the polyhedron in Lemma 4.2.16 is simply given by

$$P = \left\{ \mathbf{x} \in [0,\infty)^{\Gamma_{M,c}} \;\middle|\; \sum_{i=1}^{n} \frac{x_i}{m_i} \geq 1 \right\}.$$

This description of the polyhedron implies that for quasi-Campana pairs the b-invariant can be computed using the effective cone of X, rather than having to use the full effective cone of (X,M). This is done by replacing the canonical class with the log-canonical class, and by adding a correction factor to the b-invariant.

Proposition 4.4.10. *Let (X,M) be a smooth proper quasi-Campana pair, where X is a rationally connected variety. Assume that at least one of the following conditions hold:*

1. $\mathrm{Eff}^1(X) = \overline{\mathrm{Eff}}^1(X)$,
2. *(X,M) is a pair corresponding to Darmon points or Campana points.*

Let L be a big and nef $\mathbb{Q}$-divisor class on X. Then

$$b(K,(X,M),L) = b(K,X,K_X + D_{\mathbf{m}},L) + b'(K,(X,M),L),$$

where $b(K,X,K_X + D_{\mathbf{m}},L)$ is the codimension of the minimal face $\mathcal{F}$ of $\overline{\mathrm{Eff}}^1(X)$ containing $E = a((X,M),L)L + K_X + D_{\mathbf{m}}$, and

$$b'(K,(X,M),L) = -n' + \#I_{M,\mathcal{C}},$$

where n' is the number of divisors $D_1, \dots, D_n$ whose class does not lie in $\mathcal{F}$. The set $I_{M,\mathcal{C}}$ is the set of $(\mathbf{w},c) \in \Gamma_{M,\mathcal{C}}$ satisfying $\sum_{i=1}^n \frac{w_i}{m_i} = 1$ and $\mu((\mathbf{w},c),D) = 0$ for all *effective divisors D whose class lies in $\mathcal{F}$, where μ is given in Definition 4.1.9. In particular, if $L = -K_X - D_{\mathbf{m}}$, then*

$$b(K,(X,M),L) = \operatorname{rank}\operatorname{Pic}(X) - n + \#\left\{(\mathbf{w},c) \in \Gamma_{M,\mathcal{C}} \,\middle|\, \sum_{i=1}^{n} \frac{w_i}{m_i} = 1\right\}.$$

Remark 4.4.11. Note that if $\operatorname{Eff}^1(X)$ is generated by the classes of the divisors $D_1, \ldots, D_n$, , then an element $(\mathbf{w}, c) \in \Gamma_{M,\mathcal{C}}$ lies in $I_{M,\mathcal{C}}$ if and only if

- $\sum_{i=1}^n \frac{w_i}{m_i} = 1$, and
- $w_i = 0$ for all $i = 1, \ldots, n$ with $D_i \in \mathcal{F}$.

In particular, this is satisfied when X is a toric variety and the divisors $D_1, \ldots, D_n$ include the torus-invariant divisors on X.

Remark 4.4.12. The assumption on the effective cone is satisfied for many more varieties X including Mori dream spaces such as Fano varieties [BCHM10, Corollary 1.3.2].

Proof. By Lemma 4.2.16 and the description of the polyhedron given in Remark 4.4.9, we can without loss of generality assume that $\sum_{i=1}^n \frac{w_i}{m_i} = 1$ for all $\mathbf{w} \in \Gamma_M$. As a consequence of this assumption, we have $\operatorname{pr}_M^*(K_X + D_{\mathbf{m}}) = K_{(X,M)}$, as $\operatorname{pr}_M^* D_{\mathbf{m}} - \sum_{(\mathbf{w},c)\in\Gamma_{M,\mathcal{C}}} \tilde{D}_{\mathbf{w},c} = 0$.

We denote by $\tilde{\mathcal{F}}$ the minimal face of $\overline{\operatorname{Eff}}^1(X,M)$ containing $a((X,M),L)\operatorname{pr}_M^* L + K_{(X,M)}$. The group homomorphism $\operatorname{pr}_M^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X,M)$ induces a map $\overline{\operatorname{Eff}}^1(X) \to \overline{\operatorname{Eff}}^1(X,M)$, which we will also denote by pr_M^*. As (X,M) is proper, this map is injective by Proposition 4.2.9. This fact combined with the equality $\operatorname{pr}_M^*(K_X + D_{\mathbf{m}}) = K_{(X,M)}$ implies that an $D \in \operatorname{Eff}^1(X)$ lies in $\mathcal{F}$ if and only if $\operatorname{pr}_M^* D \in \operatorname{Eff}^1(X,M)$. For any $(\mathbf{w},c) \in \Gamma_{M,\mathcal{C}}$ such that $\mu((\mathbf{w},c),D) > 0$ for some effective divisor D whose class is contained in $\mathcal{F}$, the divisor class $[\tilde{D}_{\mathbf{w},c}]$ lies in $\tilde{\mathcal{F}}$, as $\operatorname{pr}_M^*[D_i] \in \operatorname{pr}_M^* \mathcal{F} \subset \tilde{\mathcal{F}}$.

We write $\langle\mathcal{F}\rangle$ and $\langle\tilde{\mathcal{F}}\rangle$ for the vector spaces generated by the cones, and consider the linear map

$$f\colon \operatorname{Pic}(X)_{\mathbb{R}}/\langle\mathcal{F}\rangle \to \operatorname{Pic}(X,M)_{\mathbb{R}}/\langle\tilde{\mathcal{F}}\rangle,$$

induced by pr_M^*. Since the inverse image of $\tilde{\mathcal{F}}$ under pr_M^* is $\mathcal{F}$, f is an injective map. Since

$$\dim \operatorname{Pic}(X)_{\mathbb{R}}/\langle\mathcal{F}\rangle = b(K, X, K_X + D_{\mathbf{m}}, L)$$

and

$$\dim \operatorname{Pic}(X,M)_{\mathbb{R}}/\langle\tilde{\mathcal{F}}\rangle = b(K,(X,M),L),$$

we have

$$b(K,(X,M),L) = b(K,X,K_X + D_{\mathbf{m}},L) + \dim\operatorname{coker}(f).$$

If the only solutions $\mathbf{w} \in \Gamma_M$ to $\sum_{i=1}^n \frac{w_i}{m_i} = 1$ are of the form $m_i\mathbf{e}_i$, then pr_M^* gives an isomorphism $\operatorname{Pic}(X)_{\mathbb{R}} \to \operatorname{Pic}(X,M)_{\mathbb{R}}$, so f is an isomorphism, giving

$$b(K,(X,M),L) = b(K,X,K_X + D_{\mathbf{m}},L).$$

In particular, this proves the lemma if the pair corresponds to Campana points or Darmon points.

Now we assume that $\mathrm{Eff}^1(X) = \overline{\mathrm{Eff}}^1(X)$, and we will show that the dimension of the cokernel of f is $-n' + \#I_{M,\mathcal{C}}$.

For every $i \in \{1, \dots, n\}$, we have $\mathrm{pr}_M^* D_i = \sum_{(\mathbf{w},c)\in\Gamma_{M,\mathcal{C}}} w_i \tilde{D}_{\mathbf{w},c}$, which implies

$$m_i \tilde{D}_{m\mathbf{e}_i} = - \sum_{\substack{(\mathbf{w},c)\in\Gamma_{M,\mathcal{C}} \\ \mathbf{w}\neq m\mathbf{e}_i}} w_i \tilde{D}_{\mathbf{w},c}$$

in $\mathrm{coker}(f)$. Thus, the cokernel has $\{[\tilde{D}_{\mathbf{w},c}] \mid (\mathbf{w}, c) \in I_{M,\mathcal{C}} \setminus \{m_1\mathbf{e}_1, \dots, m_n\mathbf{e}_n\}\}$ as a generating set as a vector space. We will now show that this set is a basis.

First we will show that none of these generators lie in $\tilde{\mathcal{F}}$. Consider $(\mathbf{w}, c) \in \Gamma_{M,\mathcal{C}}$ satisfying $\mu((\mathbf{w}, c), D') = 0$ for all effective divisors D' whose class is contained in $\mathcal{F}$. If $D, D_X \in \mathrm{Div}(X)_\mathbb{Q}$ are representatives of L and K_X such that $a((X, M), L)D+D_X+D_\mathbf{m}$ is effective, then the coefficient of $\tilde{D}_{(\mathbf{w},c)}$ in $\mathrm{pr}_M^*(a((X, M), L)D + D_X + D_\mathbf{m}) \in \mathrm{Div}(X, M)_\mathbb{R}$ is

$$\mu((\mathbf{w}, c), a((X, M), L)D + D_X + D_\mathbf{m}) = 0.$$

As $\mathrm{pr}_M^*(K_X + D_\mathbf{m}) = K_{(X,M)}$, this implies that the coefficient of $\tilde{D}_{\mathbf{w},c}$ is zero in every effective representative of $a((X, M), L)\, \mathrm{pr}_M^* L + K_{(X,M)}$, and we see in particular that $[\tilde{D}_{\mathbf{w},c}]$ does not lie in $\tilde{\mathcal{F}}$.

Suppose that

$$E = \sum_{(\mathbf{w},c)\in I_{M,\mathcal{C}}\setminus\{m_1\mathbf{e}_1,\dots,m_n\mathbf{e}_n\}} a_{\mathbf{w},c}\tilde{D}_{\mathbf{w},c}$$

is a $\mathbb{Q}$-divisor on (X, M) which can be written as $\mathrm{pr}_M^* E' + E_1 - E_2$ for a $\mathbb{Q}$-divisor E' on X and two effective $\mathbb{Q}$-divisors E_1, E_2 on (X, M) such that $[E_1], [E_2] \in \tilde{\mathcal{F}}$. By modifying E' if necessary, we can assume that the restriction of E_1, E_2 to $\mathrm{Div}(U)_\mathbb{Q} \times \oplus_{i=1}^n \mathbb{Q}(\tilde{D}_{m_i\mathbf{e}_i,D_i})$ is trivial. Since the restriction of E to $\mathrm{Div}(U)_\mathbb{Q} \times \oplus_{i=1}^n \mathbb{Q}(\tilde{D}_{m_i\mathbf{e}_i,D_i})$ is trivial as well, we must have $E' = 0$. For any $(\mathbf{w}, c) \in I_{M,\mathcal{C}}$ and any effective $\mathbb{Q}$-divisor D' on (X, M) whose class lies in $\tilde{\mathcal{F}}$, the coefficient of $\tilde{D}_{\mathbf{w},c}$ in D' is zero, as we have shown earlier in the proof. This implies $E_1, E_2 = 0$ so $E = 0$. Thus $\{[\tilde{D}_{\mathbf{w},c}] \mid (\mathbf{w}, c) \in I_{M,\mathcal{C}} \setminus \{m_1\mathbf{e}_1, \dots, m_n\mathbf{e}_n\}\}$ is a basis of $\mathrm{coker}(f)$, and this set contains $-n' + \#I_{M,\mathcal{C}}$ elements, so we have shown

$$b(K, (X, M), L) = b(K, X, K_X + D_\mathbf{m}, L) - n' + \#I_{M,\mathcal{C}},$$

as desired. □

Corollary 4.4.13. *Let (X, M) be a smooth proper pair corresponding to either the Campana points or the Darmon points on a Campana pair $(X, D_\mathbf{m})$, for a rationally connected variety X, and let L be big and nef $\mathbb{Q}$-divisor class on X. Then*

$$b(K, (X, M), L) = b(K, X, K_X + D_\mathbf{m}, L),$$

where $b(K, X, K_X + D_\mathbf{m}, L)$ is as in Proposition 4.4.10.

Proof. The elements in the set $I_{M,\mathcal{C}}$ in Proposition 4.4.10 correspond to the divisors $\tilde{D}_{m_i\mathbf{e}_i,D_i}$ for $i \in \{1, \dots, n\}$ such that D_i does not lie in $\mathcal{F}$. Consequently we see that $b'(K, (X, M), L) = 0$, which recovers the result. □

From Corollary 4.4.13, we see that the b-invariant in Conjecture 1.2.2 agrees with the b-invariant in the conjecture [PSTVA21, Conjecture 1.1] on Campana points of bounded height. Nevertheless, it is not directly clear whether the former conjecture implies the latter, as the conjectures impose different geometric conditions on the Campana pair. In the conjecture on Campana points, the assumption is made that the Campana pair $(X, D_{\mathbf{m}})$ is log Fano, meaning that $-(K_X + D_{\mathbf{m}})$ is ample, while the condition imposed in Conjecture 1.2.2 is that the pair (X, M) is rationally connected. The following conjecture relates the two conditions.

Conjecture 4.4.14. *Assume that the Campana pair $(X, D_{\mathbf{m}})$ is log Fano. Then any proper quasi-Campana pair (X, M) for $(X, D_{\mathbf{m}})$ is rationally connected.*

This conjecture is a related to a conjecture by Campana [Cam11b, Conjecture 9.10] on Campana rational connectedness for Campana pairs (see also [CLT24, Conjecture 1.4]), which implies Conjecture 4.4.14 for any pair (X, M) corresponding to the Campana points on $(X, D_{\mathbf{m}})$.

5. Counting $\mathcal{M}$-points on split toric varieties

In this chapter we will prove Theorem 5.2.5, which gives the asymptotic growth of the number of $\mathcal{M}$-points of bounded height on any quasi-proper toric pair over $\mathbb{Q}$. This result includes Theorem 1.2.7 as a special case, which shows that Conjecture 1.2.2 is true for a smooth proper toric pair over $\mathbb{Q}$. In this chapter, we will always take the integral model $(\mathcal{X}, \mathcal{M})$ to be the toric integral model as in Definition 3.1.1. Before stating Theorem 5.2.5, we study the geometry of toric pairs.

5.1 Geometry of toric pairs

In this chapter, we use the notation introduced in Chapter 3 for toric varieties and toric pairs. As indicated in Assumption 4.1.2, we work over a field of characteristic 0.

5.1.1 Torus-invariant divisors

On a toric variety, the intersection of a collection of torus-invariant prime divisors is a toric variety itself, and thus connected. Therefore, we identify Γ_M and $\Gamma_{M,\mathcal{C}}$, and we drop the c from our notation in the pairs $(\mathbf{m}, c)$ as the connected component c is uniquely determined by $\mathbf{m}$.

Definition 5.1.1. We write $\operatorname{Div}_T(X)$ for the torus-invariant divisors on a toric variety X. Analogously, we call an element of $\operatorname{Div}_T(X, M) := \bigoplus_{\mathbf{m}\in\Gamma_M} \mathbb{Z}(\tilde{D}_{\mathbf{m}})$ a *torus-invariant* divisor on (X, M).

Proposition 5.1.2. *Let (X, M) be a smooth toric pair. Every divisor D on (X, M) is linearly equivalent to an torus-invariant divisor on (X, M). Furthermore, if D is effective, it is linearly equivalent to an effective torus-invariant divisor. Hence the effective cone $\operatorname{Eff}^1(X, M)$ is a rational polyhedral cone generated by the torus-invariant prime divisors on (X, M).*

Proof. By [CLS11, Theorem 4.1.3] every divisor on X is linearly equivalent to a torus-invariant divisor. This directly implies that every divisor on (X, M) is linearly equivalent to a torus-invariant divisor on (X, M).

The proof of the statement for effective divisors is based on the proof of the analogous statements for divisors on X given in [CLS11, Proposition 4.3.2, Lemma 15.1.8]. For an effective divisor D on (X, M), let D' be a torus-invariant divisor linearly equivalent to it. Let

$$H^0(X, D') = \{f \in K(X)^\times \mid D' + \operatorname{pr}_M^* \operatorname{div} f \geq 0\} \cup \{0\},$$

where we write $E \geq 0$ for a divisor E on (X, M) to indicate that the divisor is effective. Then $H^0(X, D')$ is a vector space invariant under the natural torus action on $K(X)^\times$. Hence by [CLS11, Lemma 1.1.16],

$$H^0(X, D') = \bigoplus_{\substack{\mu \in N^\vee \\ D' + \mathrm{pr}_M^* \operatorname{div} \chi^\mu \geq 0}} K \cdot \chi^\mu,$$

where $\chi^\mu \in \mathcal{O}(U)^\times$ is the character determined by $\mu\colon N \to \mathbb{Z}$. Since D is effective, $H^0(X, D')$ has to be nontrivial, and thus there exists $\mu \in N^\vee$ such that $D' + \mathrm{pr}_M^* \operatorname{div}(\chi^\mu)$ is an effective torus-invariant divisor linearly equivalent to D. □

The next proposition shows that for toric pairs the Picard group can be calculated using torus-invariant divisors, analogously to the case of toric varieties.

Proposition 5.1.3. *Let (X, M) be a smooth toric pair over a field K. There is an exact sequence*

$$N^\vee \to \operatorname{Div}_T(X, M) \to \operatorname{Pic}(X, M) \to 0,$$

where $N^\vee \to \operatorname{Div}_T(X, M)$ is the composition of the map $N^\vee \to \operatorname{Div}_T(X)$ with the pullback map $\operatorname{Div}_T(X) \to \operatorname{Div}_T(X, M)$, where the first map is given by sending a character to its corresponding divisor. Furthermore, we have an exact sequence

$$0 \to N^\vee \to \operatorname{Div}_T(X, M) \to \operatorname{Pic}(X, M) \to 0$$

if and only if the lattice N_M from Definition 3.2.2 has finite index in N.

Proof. By Proposition 5.1.2, the group homomorphism $\operatorname{Div}_T(X, M) \to \operatorname{Pic}(X, M)$ is surjective. The torus-invariant principal divisors on (X, M) are exactly the pullbacks of torus-invariant principal divisors on X. Thus, the kernel of this homomorphism is the image of $N^\vee$ in $\operatorname{Div}_T(X, M)$ by [CLS11, Theorem 4.1.3].

The map $N^\vee \to \operatorname{Div}_T(X, M)$ is given by

$$\mu \mapsto \mathrm{pr}_M^* \operatorname{div}(\chi^\mu) = \sum_{\mathbf{m} \in \Gamma_M} \langle \mu, \phi(\mathbf{m}) \rangle \tilde{D}_{\mathbf{m}},$$

where ϕ is the homomorphism defined in Definition 3.2.2. This directly implies that $N^\vee \to \operatorname{Div}_T(X, M)$ is injective if and only if $\{\phi(\mathbf{m}) \mid \mathbf{m} \in \Gamma_M\}$ spans $N_\mathbb{Q}$ as a vector space, which is equivalent to N_M having finite index in N. □

Proposition 5.1.4. *Let (X, M) be a smooth toric pair over a field of characteristic 0. The canonical class of (X, M) as defined in Definition 4.2.10 is equal to*

$$K_{(X,M)} = -\sum_{\mathbf{m} \in \Gamma_M} [\tilde{D}_{\mathbf{m}}].$$

Proof. By [CLS11, Theorem 8.2.3], the canonical divisor class of a toric variety is

$$K_X = -\sum_{i=1}^n [D_i],$$

which directly implies the desired identity. □

Notation 5.1.5. We will denote the representative of $K_{(X,M)}$ introduced in Proposition 5.1.4 by

$$D_{(X,M)} = -\sum_{\mathbf{m}\in\Gamma_M} \tilde{D}_{\mathbf{m}} \in \operatorname{Div}_T(X,M).$$

In the proof of Theorem 5.2.5, we will consider divisors on toric pairs which are rigid with respect to the torus-invariant divisors, rather than to the full group of divisors.

Definition 5.1.6. Let $D \in \operatorname{Div}_T(X,M)_{\mathbb{Q}}$ be an effective $\mathbb{Q}$-divisor on a smooth toric pair (X,M). We say that D is *toric rigid* if D is the only effective torus-invariant $\mathbb{Q}$-divisor in its $\mathbb{Q}$-linear equivalence class. For a big and nef $\mathbb{Q}$-divisor L on X, we say that L is *toric adjoint rigid* with respect to (X,M) if the adjoint divisor class $a((X,M),L)\operatorname{pr}_M^* L + K_{(X,M)}$ is represented by a toric rigid effective $\mathbb{Q}$-divisor.

For many toric pairs, toric rigid divisors are just the same as rigid divisors.

Proposition 5.1.7. *Let (X,M) be a smooth toric pair. Then every rigid $\mathbb{Q}$-divisor is toric rigid. If the pullback map $\operatorname{Div}_T(X) \to \operatorname{Div}_T(X,M)$ is injective, then the converse also holds.*

On a proper toric pair, the pullback map is injective by Proposition 4.2.9. Thus, on such a pair, toric rigid divisors are the same as rigid divisors.

Proof. If an effective $\mathbb{Q}$-divisor $D \in \operatorname{Div}(X,M)_{\mathbb{Q}}$ is rigid, then Proposition 5.1.2 implies that it has to be torus-invariant, and thus toric rigid.

For the converse, we argue by proof by contrapositive. Let $D \in \operatorname{Div}_T(X,M)$ be an effective divisor which is not rigid. Then the vector space

$$H^0(X,kD) = \{f \in K(X)^\times \mid D + \operatorname{pr}_M^* \operatorname{div} f \geq 0\} \cup \{0\}$$

considered in Proposition 5.1.2 is at least two dimensional for some positive integer k. As

$$H^0(X,kD) = \bigoplus_{\substack{\mu\in N^\vee \\ kD+\operatorname{pr}_M^*\operatorname{div}\chi^\mu\geq 0}} K\cdot\chi^\mu,$$

this implies that there exist at least one nonzero $\mu \in N^\vee$ such that $kD+\operatorname{pr}_M^*\operatorname{div}\chi^\mu \geq 0$. If $\operatorname{Div}_T(X) \to \operatorname{Div}_T(X,M)$ is injective, then $\operatorname{div}\chi^\mu \neq 0$ so $D+\operatorname{div}\chi^\mu$ is an effective torus-invariant divisor linearly equivalent to D and thus D is not toric rigid. ☐

In general there may be more toric rigid divisors than rigid divisors, as the next examples show.

Example 5.1.8. Let (X,M) be the toric pair given by $X = \mathbb{P}^1$ and $\mathfrak{M} = \{(0,0)\}$, i.e., the pair corresponding to integral points for the open $\mathbb{G}_m \subset \mathbb{P}^1$. Then $\operatorname{Div}_T(X,M) = 0$ so the trivial divisor on (X,M) is toric rigid. On the other hand, $\operatorname{Pic}(\mathbb{G}_m) = 0$ so any effective divisor on (X,M) is linearly equivalent to the trivial divisor, and thus the trivial divisor is not rigid.

Example 5.1.9. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let (X,M) be the toric pair corresponding to the integral points for the open $\mathbb{G}_m \times \mathbb{P}^1 \subset X$. Then the previous example implies that any big and nef $\mathbb{Q}$-divisor on X is toric adjoint rigid with respect to (X,M).

An analogous statement is true for Hirzebruch surfaces of higher degree.

Example 5.1.10. Let $X = \mathbf{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ be the Hirzebruch surface of degree $d > 0$. Let D_2 be the unique prime divisor with self-intersection $D_2^2 = -d$, and let D_1, D_3 be the torus-invariant prime divisors intersecting the prime divisor D_2. Let (X, M) be the toric pair corresponding to the open $U' = X \setminus (D_1 \cup D_3)$. We identify $\mathrm{Pic}(X, M)$ with $\mathrm{Pic}(U')$ using the natural isomorphism as in Example 4.2.5. Under this identification, the anticanonical divisor on (X, M) is $-K_{(X,M)} = [D_2] + [D_4] \in \mathrm{Pic}(U')$, which is a nonzero effective divisor class. Since $\mathrm{Pic}(U') = \mathbb{Z}$, every divisor on U' is linearly equivalent to a rational multiple of $-K_{(X,M)}$. Consequently, any big and nef $\mathbb{Q}$-divisor on X is toric adjoint rigid with respect to (X, M).

We finish the section by noticing that toric pairs are rationally connected.

Proposition 5.1.11. *A proper toric pair (X, M) over a field K is rationally connected.*

Proof. Without loss of generality, we can assume that $\mathfrak{M}$ is a monoid. Let L be an algebraically closed field containing K and let $L(t)$ be the field of rational functions over L in the variable t. Write $(X_{L(t)}, M_{L(t)})$ for the pair over $L(t)$ obtained by base changing X and $D_1, \dots, D_n$ to $L(t)$. This pair has the obvious integral model $(X_{\mathbb{P}^1_L}, M_{\mathbb{P}^1_L})$ over $\mathbb{P}^1_L$ given by base changing (X, M) to $\mathbb{P}^1_L$. Theorem 1.1.3 implies that the pair $(X_{L(t)}, M_{L(t)})$ satisfies M_L-approximation off the place $(1 : 0) \in \mathbb{P}^1_L$. Consequently, the embedding

$$(X_{\mathbb{P}^1_L}, M_{\mathbb{P}^1_L})(\mathbb{A}^1_L) = (X, M)(\mathbb{A}^1_L) \to \prod_{v \in \Omega_{L(t)} \setminus \{(1:0)\}} (X, M)(\mathcal{O}_v)$$

has dense image. In particular $(X, M)(\mathbb{A}^1_L)$ has dense image in $(X, M)(L(\!(t)\!)) \times (X, M)(L(\!(t-1)\!))$. Since $U(L) \subset (X, M)(L(\!(t)\!))$ and $U(L) \subset (X, M)(L(\!(t-1)\!))$, this implies that any two points $p_1, p_2 \in U(L)$ are contained in the image of some function $f \in (X, M)(\mathbb{A}^1_L)$. As $\mathfrak{M}$ is a monoid and (X, M) is proper, $\mathfrak{M}$ contains $a \operatorname{mult}_{(1:0)}(f) \in \mathbb{N}^n$ for some positive integer a. Let $g \colon \mathbb{P}^1_L \to \mathbb{P}^1_L$ be the map given by $(x : y) \mapsto (x^a : y^a)$. Then $f \circ g \in (X, M)(\mathbb{P}^1_L)$ and it contains the points p_1 and p_2 in its image. This implies that (X, M) is rationally connected. $\square$

5.1.2 Fans and universal torsors for pairs

In [Sal98, Section 11], Salberger verifies Manin's conjecture for split toric varieties with the anticanonical height. In order to achieve this, he reduces the counting problem to estimating the volume of a domain $D(B) \subset Y(\mathbb{R})$ in the real locus of the universal torsor Y of the variety. Afterwards, he uses the fan Σ of X to give a partition of $D(B)$ into simpler pieces: $D(B) = \bigcup_{\sigma \in \Sigma_{\max}} D(B, \sigma)$. In order to adapt the proof of Salberger to toric pairs, we thus need appropriate analogues of the fan and of the universal torsor for toric pairs.

Notation 5.1.12. Let (X, M) be a toric pair over a field K. We let $(X, \overline{M})$ be a proper toric pair with $\overline{\mathfrak{M}} = \mathfrak{M} \cup \{d_i \mathbf{e}_i \mid m\mathbf{e}_i \notin \mathfrak{M} \text{ for all } m \in \mathbb{N}^*\}$, where $\mathbf{e}_i \in \mathbb{N}^n$

denotes the i-th basis vector and $d_1, \dots, d_n$ are positive integers such that $\Gamma_M \subset \Gamma_{\overline{M}}$. Choose a simplicial fan $\Sigma_{\overline{M}}$ refining Σ with rays

$$\Sigma_{\overline{M}}(1) = \{\rho_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_{\overline{M}}\},$$

where $\rho_{\mathbf{m}} = \mathbb{R}_{\geq 0}\phi(\mathbf{m})$. Here we recall that $\phi\colon \mathbb{N}^n \to N$ is the homomorphism given by $(m_1, \dots, m_n) \mapsto \sum_{i=1}^n m_i n_{\rho_i}$. Let $\Sigma_M \subset \Sigma_{\overline{M}}$ be the subfan given by the cones whose rays lie in $\{\rho_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_M\}$.

Similarly, for a maximal cone $\sigma \in \Sigma_{\overline{M}}$, let $(X, M(\sigma))$ be the toric pair given by

$$\overline{\mathfrak{M}} = \mathfrak{M} \cup \{d_i\mathbf{e}_i \mid \rho_i \subset \sigma(1),\, m\mathbf{e}_i \notin \mathfrak{M} \text{ for all } m \in \mathbb{N}^*\},$$

and we set $\Sigma_{M(\sigma)} \subset \Sigma_{\overline{M}}$ to be the subfan given by the cones whose rays lie in $\{\rho_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_M\} \cup \sigma(1)$.

Note that the fan $\Sigma_{\overline{M}}$ does not depend on the choice of the integers $d_1, \dots, d_n$ defining the pair $(X, \overline{M})$.

Remark 5.1.13. If (X, M) is a pair which is quasi-proper with respect to some $\mathbb{Q}$-divisor class L and the integers $d_1, \dots, d_n$ are chosen sufficiently large, we have $a((X, M), L) = a((X, \overline{M}), L)$ by Proposition 4.2.22.

Remark 5.1.14. For a toric pair (X, M) with a choice of a pair $(X, \overline{M})$ as above, there may be different non-isomorphic choices for the fan $\Sigma_{\overline{M}}$. Nevertheless, for many pairs, such as pairs corresponding to Campana points, there is only one such choice. For the purposes of this chapter, the choice of the fan is not important.

Note that $\Sigma_{\overline{M}}$ is a complete fan, since Σ is as well. Furthermore, if (X, M) is proper, then $(X, \overline{M}) = (X, M)$ so $\Sigma_M = \Sigma_{\overline{M}}$.

We will also define universal torsors of toric pairs. Consider the morphism $f\colon \mathbb{A}_K^{\Gamma_M} \to \mathbb{A}_K^n$ given by sending $(x_{\mathbf{m}})_{\mathbf{m}\in\Gamma_M}$ to $(y_1, \dots, y_n)$ where $y_i = \prod_{\mathbf{m}\in\Gamma_M} x_{\mathbf{m}}^{m_i}$ for all $i \in \{1, \dots, n\}$.

Definition 5.1.15. We define the *universal torsor* Y_M *of* (X, M) as the open toric subvariety $Y_M = f^{-1}Y$ of $\mathbb{A}_K^{\Gamma_M}$, where $Y \to X$ is the universal torsor of X as defined in Section 3.1. Let $U_M = \mathbb{G}_m^{\Gamma_M}$ be the dense torus of Y_M. The restriction of $Y_M \to X$ to $U_M \to U$ is a homomorphism of dense tori, and we write $T_M \subset Y_M$ for the kernel of this homomorphism.

We call Y_M the universal torsor of (X, M), as it plays an analogous role to the universal torsor of X in Salberger's work [Sal98].

Remark 5.1.16. By Proposition 5.1.3, the group variety T_M is isomorphic to $\mathrm{Hom}(\mathrm{Pic}(X, M), \mathbb{G}_m)$. Therefore it can be seen as an analogue of the Picard torus of a toric variety, but it need not be a torus as $\mathrm{Pic}(X, M)$ can have torsion. If (X, M) is a proper pair, then f is surjective, so the natural morphism $Y_M \to X$ is surjective as well. Furthermore, Proposition 5.1.3 implies that the restriction $U_M \to U$ is a T_M-torsor.

For toric pairs which are not proper, the morphism $Y_M \to X$ need not be dominant and the analogy with the universal torsor of a toric variety partially breaks down, as the following example shows. We still maintain the same terminology however, to avoid unnecessary case distinctions.

Example 5.1.17. Let $X = \mathbb{P}^1$ and $\mathfrak{M} = \{(0,0)\}$, i.e., the pair corresponding to integral points on $\mathbb{G}_m$. Then Y_M is just a point and the map $Y_M \to X$ is given by sending the point to $(1:1)$.

5.2 $\mathcal{M}$-points of bounded height on toric varieties

5.2.1 Heights

Let X be a smooth proper split toric variety over $\mathbb{Q}$ with fan Σ. Any $\mathbb{Q}$-divisor class L on X naturally induces a height function H_L on the torus $U \subset X$, as defined for divisor classes by Batyrev and Tschinkel in [BT95, Definition 2.1.7]. In this section, we describe this height using the description of this height given in [PS24b, Section 6.3].

We can represent a $\mathbb{Q}$-divisor class L by a $\mathbb{Q}$-divisor

$$D = a_1 D_1 + \cdots + a_n D_n,$$

for some rational numbers $a_1, \ldots, a_n \in \mathbb{Q}$. For a maximal cone $\sigma \in \Sigma$, write

$$\mu_D(\sigma) = a_1 \mu_{D_1}(\sigma) + \cdots + a_n \mu_{D_n}(\sigma),$$

where $\mu_{D_i}(\sigma) \in N^\vee$ is the unique character of U such that $\chi^{-\mu_{D_i}(\sigma)}$ generates $\mathcal{O}(D_i)$ on U_σ. Let $\sigma_1, \ldots, \sigma_k$ be the maximal cones in the fan Σ. For a ray $\rho \in \Sigma$, we write n_ρ for the corresponding ray generator. For $i = 1, \ldots, n$ and $j = 1, \ldots, k$ we define

$$l^{(i)}(\mathbf{e}_j) = a_i - \langle \mu_D(\sigma_j), n_{\rho_i} \rangle, \tag{5.2.1}$$

and

$$l^{(i)}(\mathbf{s}) = \sum_{j=1}^{k} l^{(i)}(\mathbf{e}_j) s_j$$

for $\mathbf{s} \in \mathbb{C}^k$, where we recall that $\langle \cdot, \cdot \rangle \colon N_\mathbb{Q}^\vee \times N_\mathbb{Q} \to \mathbb{Q}$ is the natural pairing.

Note that $L(\sigma_j) := \sum_{i=1}^n l^{(i)}(\mathbf{e}_j) D_i$ is $\mathbb{Q}$-linearly equivalent to L by definition. All coefficients $l^{(i)}(\mathbf{e}_j)$ are nonnegative when L is nef, as the following proposition shows.

Proposition 5.2.1. *Let L be a $\mathbb{Q}$-divisor on X. Then the following holds for any maximal cone $\sigma_j \in \Sigma$:*

1. *If $\rho_i \subset \sigma_j$, then $l^{(i)}(\mathbf{e}_j) = 0$.*
2. *If L is nef, then L is semiample and $l^{(i)}(\mathbf{e}_j) \geq 0$ for all $i = 1, \ldots, n$, so $L(\sigma_j)$ is effective.*

Proof. The first part follows directly from the definition of $l^{(i)}(\mathbf{e}_j)$. If L is nef, then it is semiample by [CLS11, Theorem 6.3.12.], and the nonnegativity of $l^{(i)}(\mathbf{e}_j) \geq 0$ is proved as in [Sal98, Proposition 8.7(a)]. □

We will now assume that L is big and nef. As in [PS24b, Section 6], we define the function

$$\mathbf{x}^{L(\sigma_j)} := \prod_{i=1}^{n} x_i^{l^{(i)}(\mathbf{e}_j)},$$

for every $j = 1, \ldots, k$ and $(x_1, \ldots, x_n) \in Y(\mathbb{Q})$. Now [PS24b, Proposition 6.10] implies that the height of a point $(x_1 : \ldots : x_n) \in X(\mathbb{Q})$ with respect to L, as defined in [PS24b, Section 6.3], is given as

$$H_L(\mathbf{x}) = \prod_{v \in \Omega_{\mathbb{Q}}} \max(|\mathbf{x}^{L(\sigma_1)}|_v, \ldots, |\mathbf{x}^{L(\sigma_k)}|_v),$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in Y(\mathbb{Q})$. Let $\mathcal{Y} \to \mathcal{X}$ be the universal torsor of $\mathcal{X}$ as introduced in Section 3.1. If we choose the coordinate representatives $(x_1, \ldots, x_n)$ to be integers representing a $\mathbb{Z}$-integral point on $\mathcal{Y}$, then the formula for the height simplifies to

$$H_L(\mathbf{x}) = \max(|\mathbf{x}^{L(\sigma_1)}|, \ldots, |\mathbf{x}^{L(\sigma_k)}|).$$

5.2.2 The leading constant

In this section we present Theorem 5.2.5, which will describe the asymptotic behaviour of the toric counting function

$$N_{(X,M),L,S}(B) = \#\{P \in (\mathcal{X}, \mathcal{M})(\mathbb{Z}[\tfrac{1}{S}]) \cap U(\mathbb{Q}) \mid H_L(P) \leq B\}$$

introduced in (1.2.1). To describe the leading constant in the asymptotic, we first need to describe the α-constant of the pair.

Definition 5.2.2. Let (X, M) be a smooth toric pair, and let L be a big and nef $\mathbb{Q}$-divisor on X which is toric adjoint rigid with respect to (X, M). Let E be the unique effective $\mathbb{Q}$-divisor on (X, M) with $\mathbb{Q}$-linear equivalence class $a((X, M), L) \operatorname{pr}_M^* L + K_{(X,M)}$, and let $(X, M^\circ) \subset (X, M)$ be the pair such that $\mathfrak{M}^\circ \setminus \{\mathbf{0}\} = \Gamma_{M^\circ}$ is the set of $\mathbf{m} \in \Gamma_M$ such that the associated divisor $\tilde{D}_{\mathbf{m}}$ is not contained in the support of E.

Let $\Lambda = \operatorname{Eff}^1(X, M^\circ)$ be the effective cone, and let $\Lambda^\vee \subset \operatorname{Pic}(X, M^\circ)^\vee_{\mathbb{R}}$ be its dual cone. Then the *α-constant* of the pair (X, M) with respect to L is

$$\alpha((X, M), L) := \frac{1}{\# \operatorname{Pic}(X, M^\circ)_{\text{torsion}}} \int_{\Lambda^\vee} e^{-\langle \operatorname{pr}_{M^\circ}^*(L), \mathbf{x} \rangle} \, \mathrm{d}\mathbf{x},$$

where the integral is taken with respect to the Lebesgue measure on $\operatorname{Pic}(X, M^\circ)^\vee_{\mathbb{R}}$, normalized by the lattice $\operatorname{Pic}(X, M^\circ)^\vee \subset \operatorname{Pic}(X, M^\circ)^\vee_{\mathbb{R}}$.

Remark 5.2.3. If (X, M) is the pair corresponding to Campana points for $(X, D_{\mathbf{m}})$, then the α-constant $\alpha((X, M), L)$ is equal to the α-constant of $(X, D_{\mathbf{m}})$ defined in [PSTVA21, Section 3.3].

Remark 5.2.4. In order to compute the constant in Theorem 5.2.5, we will use a different description of the α-constant. Let $\Lambda_1^\vee \subset \Lambda^\vee$ be the collection of all linear functions in $\operatorname{Pic}(X, M^\circ)^\vee$ which evaluate to 1 at the class L and let $(\Lambda^\vee)^\circ$ be the interior of $\Lambda^\vee$. Then $\mathbb{R}_{>0} \times \Lambda_1^\vee \to (\Lambda^\vee)^\circ$ given by $(c, f) \mapsto cf$ is an analytic isomorphism. We endow $\Lambda_1^\vee$ with the unique measure μ such that the measure on $\mathbb{R}_{>0} \times E$ corresponds to the measure on $\Lambda^\vee$ under this isomorphism, where we take the measure on $\mathbb{R}_{>0}$ to be the standard Lebesgue measure. If we write

$$\alpha_{\text{Peyre}}((X, M), L) = \frac{\operatorname{Volume}(\Lambda_1^\vee)}{\# \operatorname{Pic}(X, M^\circ)_{\text{torsion}}},$$

then a general result on cones [BT95, Proposition 2.4.4] implies

$$\alpha((X,M),L) = a((X,M),L)(b(\mathbb{Q},(X,M),L)-1)!\alpha_{\mathrm{Peyre}}((X,M),L).$$

This variant of the α-constant is used in Peyre's and Salberger's work [Pey95; Sal98] on Manin's conjecture as well as in the work of Pieropan and Schindler [PS24a] for its analogue for Campana points.

Now we can finally describe the leading constant.

Theorem 5.2.5. *Let X be a proper split toric variety over $\mathbb{Q}$, let $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ be a big and nef $\mathbb{Q}$-divisor class and let S be a positive integer. Let (X,M) be a smooth toric pair which is quasi-proper with respect to L, and let $(\mathcal{X},\mathcal{M})$ be its toric integral model. Then there exists $\theta > 0$ and a polynomial Q of degree $b(\mathbb{Q},(X,M),L)-1$ such that*

$$N_{(X,M),L,S}(B) = B^{a((X,M),L)}(Q(\log B) + O(B^{-\theta})),$$

as $B \to \infty$.

Assume that either L is adjoint rigid with respect to (X,M), or $S=1$ and L is toric adjoint rigid with respect to (X,M). Let $D = a_1D_1 + \cdots + a_nD_n$ be the unique torus-invariant $\mathbb{Q}$-divisor with $\mathbb{Q}$-linear equivalence class $a((X,M),L)L$ such that $\mathrm{pr}_M^* D + D_{(X,M)}$ is effective. The leading coefficient of Q is given by

$$C = \frac{\alpha((X,M),L)}{a((X,M),L)(b(\mathbb{Q},(X,M),L)-1)!} C_\infty \prod_{p \text{ prime}} C_p,$$

where

$$C_p = (1-p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m}\in\mathfrak{M}_{\mathrm{red}}} p^{-a_{\mathbf{m}}}$$

for all prime numbers p not dividing S, and

$$C_p = (1-p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m}\in\mathbb{N}^n_{\mathrm{red}}} p^{-a_{\mathbf{m}}}$$

for all prime numbers p dividing S, where $a_{\mathbf{m}} = a_1m_1 + \cdots + a_nm_n$.

If L is adjoint rigid, then $\#\Gamma_{M^\circ} = \dim(X) + b(\mathbb{Q},(X,M),L)$ and

$$C_\infty = 2^{\dim(X)} \sum_{\sigma\in\Sigma_{\max}} \prod_{\substack{i=1\\ \rho_i\subset\sigma}}^{n} \frac{1}{a_i}.$$

If L is toric adjoint rigid, then

$$C_\infty = 2^{\dim(X)} \sum_{\sigma\in\Sigma_{\overline{M^\circ},\max}} I(\sigma) C_\infty(\sigma),$$

where $\Sigma_{\overline{M^\circ},\max}$ is the collection of maximal cones in the fan $\Sigma_{\overline{M^\circ}}$ introduced in Notation 5.1.12, $I(\sigma)$ is the index of the subgroup $\langle [\tilde{D}_{\mathbf{m}}] \mid \phi(\mathbf{m}) \notin \sigma\rangle$ in $\mathrm{Pic}(X, M^\circ(\sigma))$ and

$$C_\infty(\sigma) = \frac{\#\,\mathrm{Pic}(X,M^\circ)_{\mathrm{torsion}}}{\#\,\mathrm{Pic}(X,M^\circ(\sigma))_{\mathrm{torsion}}} \mathrm{Volume}(Z_\sigma)\dim(Z_\sigma)!.$$

Here Z_σ is the polytope consisting of all linear functions

$$f\colon V'' = \langle [\tilde{D}_{\mathbf{m}}] \in \operatorname{Pic}(X, M^\circ(\sigma)) \mid \mathbf{m} \in \Gamma_{M^\circ(\sigma)} \setminus \Gamma_{M^\circ} \rangle \to \mathbb{R}$$

satisfying $f([\tilde{D}_{\mathbf{m}}]) \geq 0$ for all $\mathbf{m} \in \Gamma_{M^\circ(\sigma)} \setminus \Gamma_{M^\circ}$ and $f\left(\sum_{\mathbf{m}\in\Gamma_{M^\circ(\sigma)}\setminus\Gamma_{M^\circ}} [\tilde{D}_{\mathbf{m}}]\right) \leq 1$, and the volume is computed with respect to the measure ν'' such that V''/Λ'' has volume 1, where Λ'' is the image of $\operatorname{Hom}(\operatorname{Pic}(X, M^\circ(\sigma)), \mathbb{Z})$ in V''.

The proof of Theorem 5.2.5 is based on the proofs of Salberger [Sal98] and de la Bretèche [dlBre01a] for Manin's conjecture for split toric varieties with the anticanonical height. In Theorem 5.2.10 we will give a geometric interpretation for the leading coefficient of Q. In particular, this shows that for Campana points the constant agrees with the prediction in [PSTVA21, §3.3]. We illustrate the theorem by applying it to a few examples.

Example 5.2.6. In [PSTVA21, Section 3.2.1], the weak Campana points on $(\mathbb{P}^2, \frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3)$ over $\mathbb{Z}$ are considered and compared to the Campana points on the same Campana pair. The set of weak Campana points is

$$\{(x:y:z) \mid x, y, z \in \mathbb{Z}\setminus\{0\},\ \gcd(x,y,z) = 1,\ xyz \text{ is squareful}\} \subset \mathbb{P}^2(\mathbb{Q}),$$

while the set of Campana points is the subset consisting of the points $(x : y : z)$ such that each integer x, y, z is squareful. In [PSTVA21, Propostition 3.6] it is proven that the number of weak Campana points with Weil height at most B and $xyz \neq 0$ is at least $c_1 B^{3/2} \log B$ as $B \to \infty$, for some constant $c_1 > 0$. Using Theorem 5.2.5, we can compute a precise asymptotic for the number of weak Campana points of bounded height. Let $(\mathbb{P}^2_{\mathbb{Q}}, M)$ be the pair corresponding to the weak Campana points on $(\mathbb{P}^2_{\mathbb{Q}}, \frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_3)$. For the divisor class $L = [D_1] \in \operatorname{Pic}(\mathbb{P}^2_{\mathbb{Q}})$, the proof of Lemma 4.2.16 implies that the pair $(\mathbb{P}^2_{\mathbb{Q}}, M^\circ)$ as in Definition 5.2.2 is contained in the proper pair (X, M') given by $\mathfrak{M}' = \{(0,0,0), (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}$. Since $\Gamma_{M'} = \mathfrak{M}' \setminus \{(0,0,0)\}$ and

$$\operatorname{pr}^*_{M'}[D_1] = \frac{1}{3}\operatorname{pr}^*_{M'}[D_1 + D_2 + D_3] = \frac{2}{3}\sum_{\mathbf{m}\in\Gamma_{M'}} [\tilde{D}_{\mathbf{m}}] = -\frac{2}{3}K_{(X,M')},$$

the adjoint divisor of L with respect to (X, M) is 0 so we must have $(\mathbb{P}^2_{\mathbb{Q}}, M^\circ) = (\mathbb{P}^2_{\mathbb{Q}}, M')$. Furthermore, this description of $\operatorname{pr}^*_{M'}[D_1]$ also implies that $a((\mathbb{P}^2_{\mathbb{Q}}, M), [D_1]) = \frac{3}{2}$ and $b(\mathbb{Q}, (X, M), [D_1]) = \operatorname{rank}\operatorname{Pic}(X, M^\circ)$. Since D_1, D_2 and D_3 are linearly equivalent to each other, the divisor $\tilde{D}_{(1,0,1)}$ is linearly equivalent to $2\tilde{D}_{(0,2,0)} - 2\tilde{D}_{(0,0,2)} + \tilde{D}_{(1,1,0)}$ and similarly $\tilde{D}_{(0,1,1)}$ is linearly equivalent to $2\tilde{D}_{(2,0,0)} - 2\tilde{D}_{(0,0,2)} + \tilde{D}_{(1,1,0)}$. By Proposition 5.1.3 these are the only relations between torus-invariant prime divisors on (X, M°), so $\operatorname{Pic}(X, M^\circ) \cong \mathbb{Z}^4$ is freely generated by $\tilde{D}_{(2,0,0)}$, $\tilde{D}_{(0,2,0)}$, $\tilde{D}_{(0,0,2)}$ and $\tilde{D}_{(1,1,0)}$, and thus $b(\mathbb{Q}, (X, M), L) = 4$. The effective cone of $(\mathbb{P}^2_{\mathbb{Q}}, M^\circ)$ is generated by the divisors $\tilde{D}_{(2,0,0)}, \tilde{D}_{(0,2,0)}, \tilde{D}_{(0,0,2)}, \tilde{D}_{(1,1,0)}, \tilde{D}_{(1,0,1)}, \tilde{D}_{(0,1,1)}$ by Proposition 5.1.2. By subdividing the dual of the effective cone $\operatorname{Eff}^1(X, M)$ into two simplicial cones, we compute the α-constant using [Bar93, Example 2.1]:

$$\frac{\alpha((\mathbb{P}^2_{\mathbb{Q}}, M), [D_1])}{a((\mathbb{P}^2_{\mathbb{Q}}, M), [D_1])(b(\mathbb{Q}, (X, M), [D_1]) - 1)!} = \frac{1}{48}.$$

Since $a((\mathbb{P}^2_{\mathbb{Q}}, M))D_1$ is linearly equivalent to $\frac{1}{2}(D_1 + D_2 + D_3)$, we find $C_\infty = 4{\cdot}4{\cdot}3 = 48$.

Finally, for each prime,

$$C_p = \left(1 - p^{-1}\right)^6 \left(\frac{1 - p^{-3/2}}{\left(1 - p^{-1/2}\right)^3} - 3p^{-1/2}\right).$$

We conclude that

$$\#\left\{(x : y : z) \in \mathbb{P}^2(\mathbb{Q}) \;\middle|\; \begin{array}{l} x, y, z \in \mathbb{Z} \setminus \{0\},\ \gcd(x, y, z) = 1, \\ xyz \text{ is squareful, } \max(|x|, |y|, |z|) \leq B \end{array}\right\} = B^{3/2}(Q(\log B) + O(B^{-\theta}))$$

as $B \to \infty$, where $\theta > 0$ is a constant and Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}} \left(1 - p^{-1}\right)^6 \left(\frac{1 - p^{-3/2}}{\left(1 - p^{-1/2}\right)^3} - 3p^{-1/2}\right) \approx 0.862.$$

Remark 5.2.7. In the previous example, there is a more elementary method to see that there exists a constant $c > 0$ such that for any real number $B > 2$ there are at least $cB^{3/2}(\log B)^3$ tuples $(x : y : z)$ with $|x|, |y|, |z| \leq B$ and xyz squareful and nonzero, which we will now give. For every choice of pairwise coprime integers $n_1, n_2, n_3, n_4, n_5, n_6$, if we set

$$(x : y : z) := (n_1^2 n_5 n_6 : n_2^2 n_4 n_6 : n_3^2 n_4 n_5),$$

then $xyz = \prod_{i=1}^6 n_i^2$ is a squareful number. The probability that 6 positive integers less than a given bound are pairwise coprime is at least $c' = \prod_{p \text{ prime}}(1 - \frac{5}{p})(1 - \frac{1}{p})^5 > 0$. Thus the number $N(B)$ of points $(x : y : z) \in \mathbb{P}^2(\mathbb{Q})$ with xyz squareful and nonzero and furthermore $\max(|x|, |y|, |z|) \leq B$ is at least

$$c'\#\{(n_1, n_2, n_3, n_4, n_5, n_6) \in (\mathbb{N}^*)^6 \mid \max(n_1^2 n_5 n_6, n_2^2 n_4 n_6, n_3^2 n_4 n_5) \leq B\}.$$

The integer $N(B)$ is equal to the integral $c' \int \cdots \int_{\mathbf{x} \in A(B)} \mathrm{d}x_1 \ldots \mathrm{d}x_6$, where $A(B)$ is the set of all $(x_1, \ldots, x_6) \in [1, \infty)^6$ such that $\max(\lfloor x_1 \rfloor^2 \lfloor x_5 \rfloor \lfloor x_6 \rfloor, \lfloor x_2 \rfloor^2 \lfloor x_4 \rfloor \lfloor x_6 \rfloor, \lfloor x_3 \rfloor^2 \lfloor x_4 \rfloor \lfloor x_5 \rfloor) \leq B$. By using the trivial upper bound $\lfloor x_i \rfloor \leq x_i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$ together with the change of variables $y_i = \log x_i$, $N(B)$ is at least

$$c' \int \cdots \int_{\mathbf{y} \in A'(\log B)} e^{y_1 + \cdots + y_6} \,\mathrm{d}y_1 \ldots \mathrm{d}y_6,$$

where $A'(\log B)$ is the set of all $(x_1, \ldots, x_6) \in [0, \infty)^6$ such that $\max(2y_1 + y_5 + y_6, 2y_2 + y_4 + y_6, 2y_3 + y_4 + y_5) \leq \log B$. On the domain $A'(B)$ the maximum value attained by the integrand is $B^{3/2}$, which is attained on a three dimensional face $F(\log B)$ of $A'(\log B)$. Let $\mathbf{n}_1, \dots, \mathbf{n}_3 \in A'(\log B)$ be vectors generating the normal space of $F(\log B)$ with respect to the standard inner product on $\mathbb{R}^6$. Let $F'(1)$ be an open of $F(1) \cap A'(1)$ which is bounded away from the boundary of $A'(1)$. Then there exists $\epsilon > 0$ such that $V(1) := \{\mathbf{y} - a_1\mathbf{n}_1 - a_2\mathbf{n}_2 - a_3\mathbf{n}_3 \in \mathbb{R}^6 \mid \mathbf{y} \in F'(1), a_1, a_2, a_3 \in [0, \epsilon)\}$ is fully contained in $A'(1)$. Let $V(\log B) = \log B V(1)$. By the previous lower bound for $N(B)$, we see that

$$N(B) \geq c' \int \cdots \int_{\mathbf{y} \in V(\log B)} e^{y_1 + \cdots + y_6} \, \mathrm{d}y_1 \dots \mathrm{d}y_6.$$

The integral over $V(\log B)$ is equal to

$$\iiint_{\mathbf{y} \in F'(\log B)} \iiint_{a_1, a_2, a_3 \in [0, \epsilon]} B^{3/2} e^{\sum_{i=1}^{6}(-a_1 n_{1,i} - a_2 n_{2,i} - a_3 n_{3,i})} \, \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3 \, \mathrm{d}\mu,$$

where μ is the Lebesgue measure on $F'(\log B)$. This is in turn equal to

$$\mathrm{Volume}(F'(\log B)) B^{3/2}(1 + o(1)) = \mathrm{Volume}(F'(1)) B^{3/2} (\log B)^3 (1 + o(1))$$

as $B \to \infty$, which gives the desired lower bound for $N(B)$.

Using the description of the b-invariant for weak Campana points given in Proposition 4.4.10, we can also determine the asymptotic growth for the number of points on projective space for which the product of the coordinates is an m-full number, generalizing Example 5.2.6.

Example 5.2.8. Let m and n be positive integers. Then Theorem 1.2.7 implies that

$$\# \left\{ (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) \,\middle|\, \begin{array}{l} x_1, \dots, x_n \in \mathbb{Z} \setminus \{0\}, \ \gcd(x_1, \dots, x_n) = 1, \\ \prod_{i=1}^{n} x_i \text{ is } m\text{-full}, \ \max(|x_1|, \dots, |x_n|) \leq B \end{array} \right\} = B^{n/m}(Q(\log B) + O(B^{-\theta}))$$

as $B \to \infty$, where $\theta > 0$ is a constant and Q is a polynomial of degree

$$\binom{m+n-1}{n-1} - \binom{m-1}{n-1} - n.$$

As we have seen in Section 2.1.4, $N(B)$ is the number of weak Campana points on the Campana pair $\left(\mathbb{P}^{n-1}_{\mathbb{Q}}, \sum_{i=1}^{n} \left(1 - \frac{1}{m}\right) D_i\right)$ of height at most B, where the divisors $D_1, \dots, D_n$ are the coordinate hyperplanes. In particular, the log-anticanonical divisor class is given by $\sum_{i=1}^{n} \frac{1}{m}[D_i]$ and Proposition 4.4.10 implies that the b-invariant is equal to

$$-n + \#\{(a_1, \dots, a_n) \in \mathbb{N}^n \mid a_1 + \cdots + a_n = m, \ \min(a_1, \dots, a_n) = 0\}.$$

The number of ways to write m as a sum of n (nonzero) integers is $\binom{m+n-1}{n-1}$ (respectively $\binom{m-1}{n-1}$), which gives the expression for the degree of Q.

5.2.3 Geometric interpretation of the constant

Before giving the proof of Theorem 5.2.5, we give a geometric interpretation for the leading constant obtained in the theorem in the case where L is adjoint rigid with respect to (X, M), by interpreting the constant as an adelic integral. In particular we will show that the constant agrees with the prediction in [PSTVA21, §3.3]. We define the *Tamagawa constant* by

$$\tau(\mathbb{Q}, S, (X, M), L) := \int_{x \in (\mathcal{X}, \mathcal{M})(\mathbf{A}_{\mathbb{Z}[1/S]})} \frac{1}{H_{a((X,M),L)L+K_X}(x)} \,\mathrm{d}\tau_{(X,M^\circ)},$$

where $H_{a((X,M),L)L+K_X}$ is the toric height corresponding to the $\mathbb{Q}$-divisor class $a((X, M), L)L + K_X$, as defined in [BT95, Definition 2.1.7]. The measure $\tau_{(X,M^\circ)}$ is defined to be $\tau_{X,\infty} \times \prod_{p \,\mathrm{prime}} (1 - p^{-1})^{b(\mathbb{Q},(X,M),L)} \tau_{X,p}$, where the measures $\tau_{X,\infty}$ and $\tau_{X,p}$ are the local measures on $X(\mathbb{R})$ and $X(\mathbb{Q}_p)$ as in [CT10, §2.1.8] induced by the toric metric on the canonical divisor class K_X as in [BT95, Theorem 2.1.6]. Equivalently, we can write

$$\tau(\mathbb{Q}, S, (X, M), L) = \int_{x \in X(\mathbf{A}_\mathbb{Q})} \frac{\delta_{M,S}(x)}{H_{a((X,M),L)L+K_X}(x)} \,\mathrm{d}\tau_{(X,M^\circ)},$$

where $\delta_{M,S} = \prod_{\substack{p \,\mathrm{prime} \\ p \nmid S}} \delta_{M,p}$ is the product of the indicator functions $\delta_{M,p}$ of the set of $\mathcal{M}$-points in $\mathcal{X}(\mathbb{Z}_p)$ for each prime p not dividing S.

Remark 5.2.9. The factors $(1 - p^{-1})^{b((X,M),L)}$ can be interpreted as an analogue of the convergence factors considered in [CT10, Theorem 1.1]. Indeed, if we view $\mathrm{Pic}(X, M^\circ)$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module by letting the Galois group act trivially on it, then the corresponding Artin L-function is simply

$$L(s, \mathrm{Pic}(X, M^\circ)/\{\mathrm{torsion}\}) = \zeta(s)^{b((X,M),L)} = \prod_{p \,\mathrm{prime}} (1 - p^{-s})^{-b((X,M),L)}.$$

Theorem 5.2.10. *In the setting of Theorem 5.2.5, assume that L is adjoint rigid with respect to (X, M). Then the leading coefficient of Q is given by*

$$C = \frac{\alpha((X, M), L)}{a((X, M), L)(b(\mathbb{Q}, (X, M), L) - 1)!} \tau(\mathbb{Q}, S, (X, M), L).$$

In particular, if (X, M) is a pair corresponding to Campana points, then the constant is compatible with the prediction in [PSTVA21, §3.3].

Proof. We will prove the theorem by computing the the Tamagawa constant as a product over all places over $\mathbb{Q}$, which will show that it agrees with the product $C_\infty \prod_{p \,\mathrm{prime}} C_p$ in Theorem 5.2.5. The $\mathbb{Q}$-divisor class $a((X, M), L)L + K_X$ on X is represented by the $\mathbb{Q}$-divisor class $a((X, M), L)D + D_X$, where D is as in Theorem 5.2.5 and $D_X = -\sum_{i=1}^n D_i$. The Tamagawa constant is equal to the product

$$\tau(\mathbb{Q}, S, (X, M), L) = \int_{X(\mathbb{R})} \frac{1}{H_{a((X,M),L)D+D_X,\infty}(x)} \,\mathrm{d}\tau_{X,\infty}$$
$$\times \prod_{p \,\mathrm{prime}} \int_{X(\mathbb{Q}_p)} \frac{(1 - p^{-1})^{b(\mathbb{Q},(X,M),L)} \delta_{M,p}(x)}{H_{a((X,M),L)D+D_X,p}(x)} \,\mathrm{d}\tau_{X,p},$$

where $H_{a((X,M),L)L+K_X} = \prod_{v\in\Omega_{\mathbb{Q}}} H_{a((X,M),L)D+D_X,v}$ as in [BT95, Definition 2.1.5] and where we set $\delta_{M,p} = 1$ if p divides S. We will first start with the archimedean place ∞.

For a maximal cone $\sigma \in \Sigma$, let $C_\sigma(\mathbb{R}) \subset X(\mathbb{R})$ be the subset as defined in [Sal98, Notation 9.1]. As shown in the proof of [Sal98, Lemma 9.10], $C_\sigma(\mathbb{R})$ is the set of all $(x_1 : \cdots : x_n) \in X(\mathbb{R})$ such that $|x_1|, \ldots, |x_n| \leq 1$ and $x_i = 1$ for all $i \in \{1, \ldots, n\}$ with $\rho_i \not\subset \sigma$. In particular we can identify $C_\sigma(\mathbb{R})$ with $[0,1]^d$. Under this identification, the measure $\tau_{X,\infty}$ corresponds to the Lebesgue measure on $[0,1]^d$ as shown in the proof of [Sal98, Proposition 9.16]. By construction, we have $X(\mathbb{R}) = \bigcup_{\sigma\in\Sigma_{\max}} C_\sigma(\mathbb{R})$ and the proof of [Sal98, Proposition 9.16] implies that $C_\sigma(\mathbb{R}) \cap C_{\sigma'}(\mathbb{R})$ for any two maximal cones $\sigma \neq \sigma'$. Finally, for $P = (x_1 : \cdots : x_n) \in C_\sigma(\mathbb{R})$, the height is simply given $H_{a((X,M),L)D+D_X,\infty}(P) = \prod_{\substack{i=1\\ \rho_i\subset\sigma}}^n x_i^{1-a_i}$. Thus we obtain

$$\int_{x\in X(\mathbb{R})} \frac{1}{H_{a((X,M),L)D+D_X,\infty}(x)}\,\mathrm{d}\tau_X = 2^{\dim(X)} \sum_{\sigma\in\Sigma_{\max}} \prod_{\substack{i=1\\ \rho_i\subset\sigma}}^n \int_0^1 x_i^{a_i-1}\,\mathrm{d}x_i = C_\infty.$$

Let p be a prime. By [Sal98, Proposition 9.14], the measure $\tau_{X,p}$ restricts to the Haar measure on $\mathbb{A}^d(\mathbb{Z}_p) = \mathbb{Z}_p^d$ for all toric subschemes $\mathbb{A}^d \subset \mathcal{X}$. In particular it follows that for all $\mathbf{m} \in \mathfrak{M}_{\mathrm{red}}$, the set

$$V_{\mathbf{m}} = \{x \in X(\mathbb{Q}_p) \mid \mathrm{mult}_p(x) = \mathbf{m}\}$$

has volume equal to $(1-p^{-1})^{\dim(X)} p^{-\sum_{i=1}^n m_i}$. Additionally, the function $H_{a((X,M),L)D+D_X,p}$ is constant on $V_{\mathbf{m}}$, where it takes the value $p^{a_{\mathbf{m}}-\sum_{i=1}^n m_i}$. This implies

$$\int_{X(\mathbb{Q}_p)} \frac{(1-p^{-1})^{b(\mathbb{Q},(X,M),L)}\delta_{M,p}(x)}{H_{a((X,M),L)D+D_X,p}(x)}\,\mathrm{d}\tau_{X,p} = (1-p^{-1})^{\dim(X)+b(\mathbb{Q},(X,M),L)} \sum_{\mathbf{m}\in\mathfrak{M}_{\mathrm{red}}} p^{-a_{\mathbf{m}}}$$

if p does not divide S, and

$$\int_{X(\mathbb{Q}_p)} \frac{(1-p^{-1})^{b(\mathbb{Q},(X,M),L)}}{H_{a((X,M),L)D+D_X,p}(x)}\,\mathrm{d}\tau_{X,p} = (1-p^{-1})^{\dim(X)+b(\mathbb{Q},(X,M),L)} \sum_{\mathbf{m}\in\mathbb{N}^n_{\mathrm{red}}} p^{-a_{\mathbf{m}}}$$

if p divides S. In either case, this agrees with the factor C_p from Theorem 5.2.5. Thus we obtain

$$\tau(\mathbb{Q}, S, (X,M), L) = C_\infty \prod_{p\text{ prime}} C_p,$$

proving the desired identity.

The compatibility with the conjecture in [PSTVA21] follows from this identity together with the straightforward identity

$$\frac{\mathrm{d}\tau_{U',D_{\mathbf{m}}}}{H_{a((X,M),L)L+K_X+D_{\mathbf{m}}}} = \frac{\mathrm{d}\tau_{U'}}{H_{a((X,M),L)L+K_X}},$$

where U' is the complement of the support of $a((X,M),L)L + K_X + D_{\mathbf{m}}$, $\tau_{U'}$ is the Tamagawa measure as in [CT10, Definition 2.8] and $\tau_{U',D_{\mathbf{m}}} = \frac{\tau_{U'}}{H_{D_{\mathbf{m}}}}$. □

5.2.4 Part 1 of the proof of Theorem 5.2.5: upper bounds

This section is devoted to proving the following weak form of Theorem 5.2.5.

Lemma 5.2.11. *Let (X, M) be a smooth toric pair over $\mathbb{Q}$ which is quasi-proper with respect to a big and nef $\mathbb{Q}$-divisor class L on X. Then there exists a constant $\theta > 0$ and a polynomial Q of degree at most $b(\mathbb{Q}, (X, M), L) - 1$ such that*

$$N_{(X,M),L,S}(B) = B^{a((X,M),L)}(Q(\log B) + O(B^{-\theta})).$$

The proof of this lemma is based on the following Tauberian theorem by de la Bretèche.

Theorem 5.2.12. *[dlBre01b, Théorème 1] Let $f\colon \mathbb{N}^k \to \mathbb{R}$ be a nonnegative arithmetic function and let F be the associated Dirichlet series*

$$F(\mathbf{s}) = \sum_{r_1=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} \frac{f(r_1, \dots, r_k)}{r_1^{s_1} \dots r_k^{s_k}}.$$

Suppose there exists $\boldsymbol{\alpha} \in \mathbb{R}^k_{>0}$ such that F satisfies the following three properties:

(P1) For all $\mathbf{s}$ with $\mathrm{Re}(\mathbf{s}) > \boldsymbol{\alpha}$, the series $F(\mathbf{s})$ converges absolutely.

(P2) There exist finite collections $\mathscr{L} = \{l_1, \dots, l_{\tilde{n}}\}$ and $\mathscr{R}$ of linear forms with non-negative coefficients such that the function H defined by

$$H(\mathbf{s}) = F(\mathbf{s} + \boldsymbol{\alpha}) \prod_{i=1}^{\tilde{n}} l_i(\mathbf{s})$$

can be analytically continued to a holomorphic function defined on

$$\mathscr{D}(\delta_1) = \{\mathbf{s} \in \mathbb{C}^k \mid \mathrm{Re}(l(\mathbf{s})) > -\delta_1, \forall l \in \mathscr{L} \cup \mathscr{R}\},$$

for some positive constant δ_1.

(P3) There exists $\delta_2 > 0$ such that for all $\epsilon > 0, \epsilon' > 0$, the upper bound

$$|H(\mathbf{s})| \ll (1 + \|\mathrm{Im}\,(\mathbf{s})\|_1^{\epsilon}) \prod_{i=1}^{\tilde{n}} (1 + |\mathrm{Im}\,(l_i(\mathbf{s}))|)^{1-\delta_2 \min\{0, \mathrm{Re}(l_i(\mathbf{s}))\}}$$

is uniform in the domain $\mathscr{D}(\delta_1 - \epsilon') \cap \{\mathbf{s} \in \mathbb{C}^k | \mathrm{Re}(\mathbf{s}) < (1, \dots, 1)\}$.

Then there exists a polynomial Q of degree at most $\tilde{n} - \mathrm{rank}(l_1, \dots, l_{\tilde{n}})$ such that

$$S(B) = \sum_{r_1=1}^{B} \cdots \sum_{r_k=1}^{B} f(r_1, \dots, r_k) = B^{\sum_{i=1}^k \alpha_i}(Q(\log B) + O(B^{-\theta}))$$

for all $B \geq 1$, where $\theta > 0$ is a constant depending on $\boldsymbol{\alpha}, \delta_1, \delta_2, \mathscr{L}, \mathscr{R}$.

Remark 5.2.13. Theorem 5.2.12 has a typo in its original formulation [dlBre01b, Théorème 1], as it requires the the upper bound in (P3) to be satisfied on the whole of $\mathscr{D}(\delta_1 - \epsilon')$. This is a stronger assumption than necessary and desired, as it would imply that H is bounded on $\mathbb{R}^k_{>0}$, which would exclude the original application of the theorem in [dlBre01a].

Dirichlet series

Since the height function satisfies $H_{tL}(\mathbf{x}) = H_L(\mathbf{x})^t$ for any $t \in \mathbb{Q}_{>0}$, we assume without loss of generality that $L \in \operatorname{Pic}(X)$. As L is integral, the function $\mathbf{x}^{L(\sigma_j)}$ is simply a monomial, and thus takes integer values on integer inputs.

For $(r_1, \dots, r_k) \in \mathbb{N}^k$, we define $f(r_1, \dots, r_k)$ to be

$$\#\left\{\mathbf{d} \in (\mathbb{N}^*)^n \,\middle|\, \begin{array}{l} \mathbf{d}^{L(\sigma_j)} = r_j \text{ for all } j = 1, \dots, k; \quad \gcd(\mathbf{d}^{\hat{\sigma}_1}, \dots, \mathbf{d}^{\hat{\sigma}_k}) = 1; \\ \operatorname{mult}_p(\mathbf{d}) \in \mathfrak{M} \text{ for all prime numbers } p \nmid S \end{array}\right\},$$

where $\operatorname{mult}_p(\mathbf{d}) = (v_p(x_1), \dots, v_p(x_n))$ is the tuple given by the p-adic valuations of the components of $\mathbf{d}$, and the monomials

$$\mathbf{x}^{\hat{\sigma}} = \prod_{\substack{i=1 \\ \rho_i \not\subset \sigma}}^{n} x_i$$

are defined as in Section 3.1.

In this notation, $N_{(X,M),L,S}(B) = 2^{\dim(X)} S(B)$, where

$$S(B) := \sum_{r_1=1}^{B} \cdots \sum_{r_k=1}^{B} f(r_1, \dots, r_k).$$

Here the factor of $2^{\dim(X)}$ accounts for the fact that f only counts the points that can be described using positive coordinates. We estimate the sum $S(B)$ by considering the multiple Dirichlet series

$$F(\mathbf{s}) := \sum_{r_1=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} \frac{f(r_1, \dots r_k)}{r_1^{s_1} \dots r_k^{s_k}}.$$

In order to apply Theorem 5.2.12, we will rewrite F as a multiple Dirichlet series of a multiplicative function. Let $\chi\colon (\mathbb{N}^*)^n \to \{0, 1\}$ be the characteristic function of the set

$$\{\mathbf{d} \in (\mathbb{N}^*)^n \mid \gcd(\mathbf{d}^{\hat{\sigma}_1}, \dots, \mathbf{d}^{\hat{\sigma}_k}) = 1, \operatorname{mult}_p(\mathbf{d}) \in \mathfrak{M} \text{ for all primes } p \nmid S\},$$

so

$$\frac{f(r_1, \dots r_k)}{r_1^{s_1} \dots r_k^{s_k}} = \sum_{\substack{\mathbf{d} \in \mathbb{N}^n \\ \mathbf{d}^{L(\sigma_j)} = r_j \, \forall j \in \{1, \dots, k\}}} \frac{\chi(\mathbf{d})}{d_1^{l^{(1)}(\mathbf{s})} \dots d_n^{l^{(n)}(\mathbf{s})}}.$$

This equality implies

$$F(\mathbf{s}) = \sum_{d_1=1}^{\infty} \cdots \sum_{d_n=1}^{\infty} \frac{\chi(\mathbf{d})}{d_1^{l^{(1)}(\mathbf{s})} \dots d_n^{l^{(n)}(\mathbf{s})}}.$$

Since the condition $\gcd(\mathbf{d}^{\hat{\sigma}_1}, \dots, \mathbf{d}^{\hat{\sigma}_k}) = 1$ only depends on the valuations $\operatorname{mult}_p(\mathbf{d})$ at each prime, the function χ is multiplicative in the sense that

$$\chi(\mathbf{d}\mathbf{d}') = \chi(\mathbf{d})\chi(\mathbf{d}')$$

for all $\mathbf{d}, \mathbf{d}' \in (\mathbb{N}^*)^n$ satisfying $\gcd(d_1 \dots d_n, d_1' \dots d_n') = 1$.

Since χ is multiplicative, we can write

$$F(\mathbf{s}) = \prod_{p \text{ prime}} F_p(\mathbf{s}),$$

where for primes numbers p not dividing S,

$$F_p(\mathbf{s}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\chi(p^{m_1}, \dots, p^{m_n})}{p^{l_{\mathbf{m}}(\mathbf{s})}} = \sum_{\mathbf{m} \in \mathfrak{M}_{\text{red}}} p^{-l_{\mathbf{m}}(\mathbf{s})},$$

and similarly for prime numbers p dividing S,

$$F_p(\mathbf{s}) = \sum_{\mathbf{m} \in \mathbb{N}^n_{\text{red}}} p^{-l_{\mathbf{m}}(\mathbf{s})}.$$

Here we write $l_{\mathbf{m}} = m_1 l^{(1)} + \dots + m_n l^{(n)}$ for $\mathbf{m} \in \mathbb{N}^n$. Note that for all $i \in \{1, \dots, n\}$ and $\mathbf{s} \in \mathbb{Q}^k$, the value $l^{(i)}(\mathbf{s})$ is the coefficient of D_i in $\sum_{j=1}^k s_j L(\sigma_j)$. This can be seen as a direct consequence of the definition of the linear forms $l^{(1)}, \dots, l^{(n)}$ as given in (5.2.1). For $\mathbf{m} \in \Gamma_M$, this gives a simple geometric interpretation for the linear form $l_{\mathbf{m}}$: for $\mathbf{s} \in \mathbb{Q}^k$, the value $l_{\mathbf{m}}(\mathbf{s})$ is the coefficient of $\tilde{D}_{\mathbf{m}}$ in $\mathrm{pr}_M^* \sum_{j=1}^k s_j L(\sigma_j)$.

Using the product formula $F(\mathbf{s}) = \prod_{p \text{ prime}} F_p(\mathbf{s})$, we determine an open set on which $F(\mathbf{s})$ converges.

Proposition 5.2.14. *The series $F_p(\mathbf{s})$ converges absolutely in the region*

$$V = \{\mathbf{s} \in \mathbb{C}^k \mid \mathrm{Re}(l^{(i)}(\mathbf{s})) > 0 \textit{ for all } i = 1, \dots, n\}.$$

Furthermore, for any $\epsilon > 0$ and any prime number $p \notin S$,

$$F_p(\mathbf{s}) = 1 + O(p^{-1-\epsilon})$$

in the region $V \cap \{\mathbf{s} \in \mathbb{C}^k \mid \mathrm{Re}(l_{\mathbf{m}}(\mathbf{s})) > 1 + \epsilon$ for all $\mathbf{m} \in \mathfrak{M} \setminus \{\mathbf{0}\}\}$. Here the implicit constant depends on ϵ but not on p.

Consequently, the series $F(\mathbf{s})$ converges in the region

$$V \cap \{\mathbf{s} \in \mathbb{C}^k \mid \mathrm{Re}(l_{\mathbf{m}}(\mathbf{s})) > 1 \textit{ for all } \mathbf{m} \in \mathfrak{M} \setminus \{\mathbf{0}\}\}.$$

Proof. The series

$$\sum_{\mathbf{m} \in \mathbb{N}^n} p^{-\mathrm{Re}(l_{\mathbf{m}}(\mathbf{s}))} = \prod_{i=1}^{n} \sum_{m_i=1}^{\infty} \left(p^{-\mathrm{Re}(l^{(i)}(\mathbf{s}))}\right)^{m_i}$$

converges to $\prod_{i=1}^n \left(1 - p^{-\mathrm{Re}(l^{(i)}(\mathbf{s}))}\right)^{-1}$ for all $\mathbf{s} \in V$, as it is simply a product of convergent geometric series. By comparing $F_p(\mathbf{s})$ with this series we directly obtain the absolute convergence of $F_p(\mathbf{s})$ for $\mathbf{s} \in V$. Let $\mathbf{m}_1, \dots, \mathbf{m}_t$ be the minimal nonzero elements in $\mathfrak{M}$. Every nonzero element $\mathbf{m} \in \mathfrak{M}$ can be written as $\mathbf{m}_i + \mathbf{m}'$ for some $i \in \{1, \dots, t\}$ and $\mathbf{m}' \in \mathbb{N}^n$, so $|F_p(\mathbf{s}) - 1|$ is dominated by the series

$$\left(p^{-\mathrm{Re}(l_{\mathbf{m}_1}(\mathbf{s}))} + \dots + p^{-\mathrm{Re}(l_{\mathbf{m}_t}(\mathbf{s}))}\right) \sum_{\mathbf{m} \in \mathbb{N}^n} p^{-\mathrm{Re}(l_{\mathbf{m}}(\mathbf{s}))}.$$

As $\sum_{\mathbf{m}\in\mathbb{N}^n} p^{-\operatorname{Re}(l_{\mathbf{m}}(\mathbf{s}))}$ is bounded on $V \cap \{\mathbf{s} \in \mathbb{C}^k \mid \operatorname{Re}(l_{\mathbf{m}}(\mathbf{s})) > 1$ for all $\mathbf{m} \in \mathfrak{M} \setminus \{\mathbf{0}\}\}$, $F_p(\mathbf{s})$ satisfies the estimate $F_p(\mathbf{s}) = 1 + O(p^{-1-\epsilon})$ on this region. In turn, the estimate for $p \notin S$ implies that the product $F(\mathbf{s}) = \prod_{p \text{ prime}} F_p(\mathbf{s})$ converges whenever $\mathbf{s}$ lies in

$$V \cap \{\mathbf{s} \in \mathbb{C}^k \mid \operatorname{Re}(l_{\mathbf{m}}(\mathbf{s})) > 1 \text{ for all } \mathbf{m} \in \mathfrak{M} \setminus \{\mathbf{0}\}\}. \qquad \square$$

By Proposition 5.2.14, if a tuple $\boldsymbol{\alpha} \in \mathbb{R}^k_{\geq 0}$ satisfies $\operatorname{Re}(l_{\mathbf{m}}(\boldsymbol{\alpha})) \geq 1$ for all $\mathbf{m} \in \mathfrak{M} \setminus \{\mathbf{0}\}$ as well as $\operatorname{Re}(l^{(i)}(\boldsymbol{\alpha})) > 0$ for all $i = 1, \dots, n$, then condition (P1) from Theorem 5.2.12 will be satisfied. However, in order to find a good bound for $N_{(X,M),L,S}(B)$ using Theorem 5.2.12, we need to minimize the sum $\sum_{j=1}^k \alpha_j$.

Choice of $\boldsymbol{\alpha}$

Let $\mathscr{P}_M$ be the following linear program: minimize the function $\sum_{j=1}^k \alpha_j$, under the conditions $\alpha_j \geq 0$ for all $j = 1, \dots, k$ and $l_{\mathbf{m}}(\boldsymbol{\alpha}) \geq 1$ for all $\mathbf{m} \in \mathfrak{M}$. Since $l_{\mathbf{m}} = m_1 l^{(1)} + \cdots + m_n l^{(n)}$, the latter condition is equivalent to the condition

$$\sum_{j=1}^k \alpha_j \operatorname{pr}_M^* L(\sigma_j) + D_{(X,M)} \geq 0.$$

This condition in turn implies that $(\sum_{j=1}^k \alpha_j) \operatorname{pr}_M^* L + K_{(X,M)} \in \operatorname{Eff}^1(X, M)$, so any solution $\boldsymbol{\alpha}$ to $\mathscr{P}_M$ has to satisfy $\sum_{j=1}^k \alpha_j \geq a((X,M),L)$.

We will use the following proposition to show that the equality $\sum_{j=1}^k \alpha_j = a((X,M),L)$ can be achieved.

Proposition 5.2.15. *Let $D = a_1 D_1 + \cdots + a_n D_n$ be a torus-invariant $\mathbb{Q}$-divisor representing $L \in \operatorname{Pic}(X)_{\mathbb{Q}}$. Then*

$$\sum_{\sigma\in\Sigma_{\max}} \left(\prod_{\rho_i \notin \sigma} a_i \right) \mu_D(\sigma) = 0,$$

where we recall the notation $\mu_D = a_1 \mu_{D_1} + \cdots + a_n \mu_{D_n}$. Thus

$$\sum_{\sigma\in\Sigma_{\max}} L(\sigma) \prod_{\rho_i \notin \sigma} a_i = \left(\sum_{\sigma\in\Sigma_{\max}} \prod_{\rho_i \notin \sigma} a_i \right) D.$$

Proof. Let $\sigma \in \Sigma$ be a maximal cone in the fan of X, and order the rays such that $\rho_1, \dots \rho_d$ are the rays in σ, where $d = \dim X$. Then $n_{\rho_1}, \dots, n_{\rho_d}$ forms a lattice basis of N, since X is smooth. Denote the corresponding dual basis by $n^*_{\rho_1}, \dots, n^*_{\rho_d} \in N^\vee$. Then we see by definition that

$$\mu_D(\sigma) = \sum_{i=1}^d a_i n^*_{\rho_i}.$$

Recall that a linear form $N \to \mathbb{Z}$ is irreducible if it is not a positive multiple of another linear form. For τ a facet of σ, we write $u_{\sigma,\tau} \in N^\vee$ for the unique irreducible linear form which is zero on τ and positive on $\sigma \setminus \tau$. This linear form is simply given as $u_{\sigma,\tau} = n^*_{\rho_i}$, where ρ_i is the unique ray in $\sigma(1)\setminus\tau(1)$. We also write $a_{\sigma,\tau} = a_i$. By [Sal98, Lemma 8.9(i)] there is a unique maximal cone $\sigma' \neq \sigma$ containing τ, which satisfies $\tau = \sigma' \cap \sigma$. Since $u_{\sigma',\tau}$ is also irreducible and zero on τ, we see $u_{\sigma',\tau} = \pm n^*_{\rho_i}$. Since there are $n_1 \in \sigma \setminus \tau$, $n_2 \in \sigma' \setminus \tau$ such that $n_1 + n_2 \in \tau$, we see $u_{\sigma',\tau} = -n^*_{\rho_i} = -u_{\sigma,\tau}$.

Therefore, we get

$$\sum_{\sigma \in \Sigma_{max}} \left(\prod_{\substack{i=1 \\ \rho_i \notin \sigma}}^{n} a_i \right) \mu_D(\sigma) = \sum_{\sigma \in \Sigma_{max}} \left(\prod_{\substack{i=1 \\ \rho_i \notin \sigma}}^{n} a_i \right) \sum_{\tau \text{ facet of } \sigma} a_{\sigma,\tau} u_{\sigma,\tau}$$
$$= \sum_{\tau \text{ facet of } \Sigma} \sum_{\substack{\sigma \in \Sigma_{\max} \\ \tau \subset \sigma}} \left(\prod_{\substack{i=1 \\ \rho_i \notin \sigma}} a_i \right) u_{\sigma,\tau} = 0. \qquad \square$$

Corollary 5.2.16. *The $\mathbb{Q}$-divisor class $a((X,M),L)L$ can be represented by a $\mathbb{Q}$-divisor $D = a_1 D_1 + \cdots + a_n D_n$ on X such that $a_1, \ldots, a_n > 0$ and such that $a((X,M),L) \operatorname{pr}_M^* D + D_{(X,M)}$ is effective. Furthermore, for any such D, there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^k_{>0}$ such that $D = \sum_{j=1}^k \alpha_j L(\sigma_j)$ and $\sum_{j=1}^k \alpha_j = a((X,M),L)$.*

Proof. Let $(X, \overline{M})$ be a proper pair as in Definition 4.2.21. Let $D = a_1 D_1 + \cdots + a_n D_n$ be a $\mathbb{Q}$-divisor such that $a((X,\overline{M}),L) \operatorname{pr}_{\overline{M}}^* D + D_{(X,\overline{M})}$ is an effective $\mathbb{Q}$-divisor. As $(X,\overline{M})$ is proper, this implies $a_1, \ldots, a_n > 0$ as $D_{(X,\overline{M})} = -\sum_{\mathbf{m} \in \Gamma_{\overline{M}}} \tilde{D}_{\mathbf{m}}$ is a divisor representing the canonical class $K_{(X,\overline{M})}$. Since $a((X,\overline{M}),L) = a((X,M),L)$, the divisor $a((X,M),L) \operatorname{pr}_M^* D + D_{(X,M)}$ is simply the restriction of $a((X,\overline{M}),L) \operatorname{pr}_{\overline{M}}^* D + D_{(X,\overline{M})}$ to $\operatorname{Div}(X,M)_{\mathbb{Q}}$. Now Proposition 5.2.15 implies that if we take

$$\beta_j = \frac{\prod_{\rho_i \notin \sigma_j} a_i}{\sum_{\sigma \in \Sigma_{\max}} \prod_{\rho_i \notin \sigma} a_i},$$

then $D = \sum_{j=1}^k \beta_j L(\sigma_j)$.

By setting $\boldsymbol{\alpha} := a((X,M),L)\boldsymbol{\beta}$, we find that

$$\sum_{j=1}^{k} \alpha_j \operatorname{pr}_M^* L(\sigma_j) + D_{(X,M)} = a((X,M),L) \operatorname{pr}_M^* D + D_{(X,M)}$$

is effective, $\sum_{j=1}^k \alpha_j = a((X,M),L)$ and $\alpha_j > 0$ for all $j = 1, \ldots, k$. $\square$

Choice of linear forms

In this section we will choose the set $\mathscr{L}$ of linear forms, and we verify that conditions (P2) and (P3) in Theorem 5.2.12 are satisfied with this choice.

Assumption 5.2.17. Without loss of generality, we assume that we have chosen the representative $D = a_1 D_1 + \cdots + a_n D_n$ of L such that the $\mathbb{Q}$-divisor

$a((X,M),L)\operatorname{pr}_M^* D + D_{(X,M)} = \sum_{\mathbf{m}\in\Gamma_M} \tilde{a}_{\mathbf{m}}\tilde{D}_{\mathbf{m}}$ is effective and has maximal support: we assume $\tilde{a}_{\mathbf{m}} > 0$ for as many $\mathbf{m}\in\Gamma_M$ as possible for such a representative of L.

Let $(X, M^\circ) \subset (X,M)$ be the toric pair as in Definition 5.2.2 and let $\boldsymbol{\alpha}$ be as in Corollary 5.2.16. For $\mathbf{m}\in\Gamma_M$, the coefficient of $\tilde{D}_{\mathbf{m}}$ in $a((X,M),L)\operatorname{pr}_M^* D + D_{(X,M)}$ is given by $l_{\mathbf{m}}(\boldsymbol{\alpha}) - 1$. Thus, by Assumption 5.2.17, $\mathbf{m}\in\Gamma_M$ lies in $\mathfrak{M}^\circ$ if and only if $l_{\mathbf{m}}(\boldsymbol{\alpha}) = 1$. Let

$$\mathscr{L} = \{l_{\mathbf{m}} \mid \mathbf{m}\in\Gamma_{M^\circ}\} \text{ and } \mathscr{R} = \{l^{(1)},\dots,l^{(n)}\}.$$

Since $a_i > 0$ for all $i = 1,\dots,n$, $l^{(i)}(\boldsymbol{\alpha}) = \frac{a_i}{a((X,M),L)} > 0$. Furthermore, if $\mathbf{m},\mathbf{m}'\in\mathbb{N}^n$ satisfy $\mathbf{m} < \mathbf{m}'$ using the natural partial order on $\mathbb{N}^n$, then $\operatorname{Re}(l_{\mathbf{m}}(\mathbf{s})) < \operatorname{Re}(l_{\mathbf{m}'}(\mathbf{s}))$ for all $\mathbf{s}\in\mathbb{C}^k$ satisfying $\operatorname{Re}(l^{(i)}(\mathbf{s})) > 0$. As $\mathfrak{M}_{\mathrm{red,mon}} \setminus \mathfrak{M}^\circ$ has a finite number of minimal elements in this ordering on $\mathbb{N}^n$, the continuity of the linear forms $l_{\mathbf{m}}$ and $l^{(i)}$ implies that there exist $\frac{1}{4} > \delta_1 > 0, \epsilon > 0$ such that for all

$$\mathbf{s}\in\mathscr{D}(\delta_1) = \{\mathbf{s}\in\mathbb{C}^k \mid \operatorname{Re}(l(\mathbf{s})) > -\delta_1\, \forall l\in\mathscr{L}\cup\mathscr{R}\},$$

we have that $\operatorname{Re}(l_{\mathbf{m}}(\boldsymbol{\alpha}+\mathbf{s})) > 1+\epsilon$ for all $\mathbf{m}\in\mathfrak{M}_{\mathrm{red,mon}}\setminus\mathfrak{M}^\circ$ and $\operatorname{Re}(l^{(i)}(\boldsymbol{\alpha}+\mathbf{s})) > \epsilon$.

Consider the function

$$H(\mathbf{s}) = F(\mathbf{s}+\boldsymbol{\alpha}) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} l_{\mathbf{m}}(\mathbf{s}),$$

and write

$$\frac{F(\mathbf{s})}{\prod_{\mathbf{m}\in\Gamma_{M^\circ}}\zeta(l_{\mathbf{m}}(\mathbf{s}))} = \prod_{p \text{ prime}} G_p(\mathbf{s}),$$

where

$$G_p(\mathbf{s}) = \left(\sum_{\mathbf{m}\in\mathfrak{M}_{\mathrm{red}}} p^{-l_{\mathbf{m}}(\mathbf{s})}\right) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} (1 - p^{-l_{\mathbf{m}}(\mathbf{s})})$$

for all prime numbers p not dividing S.

The product

$$\left(\sum_{\mathbf{m}\in\mathfrak{M}^\circ} p^{-l_{\mathbf{m}}(\mathbf{s})}\right) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} (1 - p^{-l_{\mathbf{m}}(\mathbf{s})})$$

is a finite sum of the form $1 + \sum_{\mathbf{m}\in I} c_{\mathbf{m}} p^{-l_{\mathbf{m}}(\mathbf{s})}$, where $I \subset \mathfrak{M}_{\mathrm{red,mon}}$ is a finite set disjoint from Γ_{M°. In particular we see that the absolute value of $\left(\sum_{\mathbf{m}\in\mathfrak{M}^\circ} p^{-l_{\mathbf{m}}(\boldsymbol{\alpha}+\mathbf{s})}\right)\prod_{\mathbf{m}\in\Gamma_{M^\circ}}(1 - p^{-l_{\mathbf{m}}(\boldsymbol{\alpha}+\mathbf{s})}) - 1$ is bounded by $\#I \cdot p^{-1-\epsilon}$ for all $\mathbf{s}\in\mathscr{D}(\delta_1)$. By writing

$$G_p(\mathbf{s}) = \left(\sum_{\mathbf{m}\in\mathfrak{M}^\circ} p^{-l_{\mathbf{m}}(\mathbf{s})} + \sum_{\mathbf{m}\in\mathfrak{M}_{\mathrm{red}}\setminus\mathfrak{M}^\circ} p^{-l_{\mathbf{m}}(\mathbf{s})}\right) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} (1 - p^{-l_{\mathbf{m}}(\mathbf{s})}),$$

we find that the function $G_p(\mathbf{s})$ satisfies $G_p(\mathbf{s}+\boldsymbol{\alpha}) = 1 + O(p^{-1-\epsilon})$ for all $\mathbf{s}\in\mathscr{D}(\delta_1)$ and for all prime numbers p not dividing S, where the implied constant depends on δ_1

but is independent of the prime p. Thus the product $G(\mathbf{s}) = \prod_{p \text{ prime}} G_p(\mathbf{s})$ converges to a bounded holomorphic function on $\boldsymbol{\alpha} + \mathscr{D}(\delta_1)$. Since $s\zeta(s+1)$ is a holomorphic function, this implies that $H(\mathbf{s})$ can be analytically continued to a function on $\mathscr{D}(\delta_1)$ and therefore condition (P2) of Theorem 5.2.12 is satisfied.

Now, as in de la Bretèche's work [dlBre01a, §4.3], we will use the upper bound

$$z\zeta(z+1) \ll (\operatorname{Im} z + 1)^{1-\min(\operatorname{Re} z,0)/3+\epsilon}, \quad T \geq \operatorname{Re}(z) \geq -\tfrac{1}{2},$$

valid for all $\epsilon > 0$ and $T > 0$, which follows from [Ten15, Theorem II.3.8]. By shrinking δ_1 if necessary, we can assume that G extends to a holomorphic function on the topological closure of $\boldsymbol{\alpha} + \mathscr{D}(\delta_1)$. Now since $G(\mathbf{s})$ is bounded on $\boldsymbol{\alpha} + \mathscr{D}(\delta_1)$, this implies that condition (P3) in Theorem 5.2.12 is satisfied with $\delta_2 = 1/3$.

Thus all conditions of Theorem 5.2.12 are satisfied with the choices made above for $\boldsymbol{\alpha}, \delta_1, \delta_2, \mathscr{L}$ and $\mathscr{R}$.

Determining the rank

To finish the proof of Lemma 5.2.11, we need to determine the rank of $\mathscr{L}$. We will first compute the rank of the matrix given by the linear forms $l^{(1)}, \dots, l^{(n)}$. For vectors $l_1, \dots, l_n \in \mathbb{R}^k$, we write $(l_1, \dots, l_n)$ for the matrix with rows $l_1, \dots, l_n$.

Proposition 5.2.18. *If L is a big and nef $\mathbb{Q}$-divisor class, then the rank of the matrix $(l^{(1)}, \dots, l^{(n)})$ is* $\dim(X)+1$. *Consequently, any $\mathbb{Q}$-divisor in* $\operatorname{Div}(X)_{\mathbb{Q}}$ *that is $\mathbb{Q}$-linearly equivalent to* 0 *lies in*

$$V = \left\langle \sum_{j=1}^{k} y_j \operatorname{pr}_M^* L(\sigma_j) \,\middle|\, (y_1, \dots, y_k) \in \mathbb{Q}^k,\ \sum_{j=1}^{k} y_j = 0 \right\rangle.$$

Proof. Without loss of generality we can assume that L is a divisor class, rather than just a $\mathbb{Q}$-divisor class. We represent L by the divisor $D' := L(\sigma_1) = a'_1 D_1 + \cdots + a'_n D_n$. Then we have $\mu_{D'}(\sigma_1) = 0$, so the first column of the matrix $(l^{(1)}, \dots, l^{(n)})$ is just $(a'_1, \dots, a'_n)$. Since L is a nonzero divisor class, the divisor $L(\sigma_1)$ does not lie in the linear span of the divisors $L(\sigma_1) - L(\sigma_2), \dots, L(\sigma_1) - L(\sigma_k)$. Therefore

$$\operatorname{rank}(l^{(1)}, \dots, l^{(n)}) = \operatorname{rank}(A) + 1,$$

where A is the matrix such that the coefficient in position (i,j) is $\langle \mu_{D'}(\sigma_j), n_{\rho_i} \rangle$. Since a nef divisor on a toric variety is globally generated by [CLS11, Theorem 6.3.12], we see by [CLS11, Theorem 6.1.7] that $\operatorname{rank}(A) = \dim P_L$, where P_L is the polyhedron associated to L as defined in [CLS11, §4.3]. By [CLS11, Lemma 9.3.9] we also have $\dim P_L = \dim(X)$, as L is big. The vector space V is contained in the kernel of $\operatorname{Div}_T(X)_{\mathbb{Q}} \to \operatorname{Pic}(X)_{\mathbb{Q}}$. By [CLS11, Theorem 4.2.1.] this kernel has dimension $\dim(X)$, but $\operatorname{rank}(l^{(1)}, \dots, l^{(n)}) = \dim(X) + 1$ implies V has dimension $\dim(X)$ as well, so V is equal to the kernel. □

We view $\mathscr{L}$ as a linear map $\mathbb{Q}^k \to \mathbb{Q}^{\Gamma_{M^\circ}}$, so that $\#\Gamma_{M^\circ} - \operatorname{rank}(\mathscr{L}) = \operatorname{rank} \operatorname{coker} \mathscr{L}$.

Proposition 5.2.19. *The rank of* $\operatorname{coker} \mathscr{L}$ *is equal to* $b(\mathbb{Q}, (X, M), L) - 1$.

Proof. The cokernel of $\mathscr{L}$ is the dual space of the kernel of the dual map $\mathscr{L}^\vee\colon \mathbb{Q}^{\Gamma_{M^\circ}} \to \mathbb{Q}^k$. This kernel is

$$\left\{ \mathbf{x} \in \mathbb{Q}^{\Gamma_{M^\circ}} \;\middle|\; \sum_{\mathbf{m}\in\Gamma_{M^\circ}} l_{\mathbf{m}}(\mathbf{e}_j)x_{\mathbf{m}} = 0 \text{ for all } j \in \{1,\dots,k\} \right\}.$$

Recall that, for all $j \in \{1,\dots,k\}$, $L(\sigma_j) = \sum_{i=1}^n l^{(i)}(\mathbf{e}_j)D_i$ by the defining formula (5.2.1) for $l^{(i)}(\mathbf{e}_j)$, so $\operatorname{pr}_{M^\circ}^* L(\sigma_j) = \sum_{\mathbf{m}\in\Gamma_{M^\circ}} l_{\mathbf{m}}(\mathbf{e}_j)\tilde{D}_{\mathbf{m}}$. This implies that the kernel is isomorphic to

$$\{\mathbf{x} \in \operatorname{Div}(X, M^\circ)^\vee_{\mathbb{Q}} \mid \mathbf{x}(\operatorname{pr}_{M^\circ}^* L(\sigma_j)) = 0 \,\forall j \in \{1,\dots,k\}\}.$$

Every such function is zero on torus-invariant principal divisors, as Proposition 5.2.18 implies that these are the divisors of the form $\sum_{j=1}^k y_j \operatorname{pr}_M^* L(\sigma_j)$ for $(y_1,\dots,y_k) \in \mathbb{Q}^k$ satisfying $\sum_{j=1}^k y_j = 0$. This implies that the kernel is naturally identified with

$$\{\mathbf{x} \in \operatorname{Pic}(X, M^\circ)^\vee_{\mathbb{Q}} \mid \mathbf{x}(\operatorname{pr}_{M^\circ}^* L) = 0\}.$$

As (X, M°) is quasi-proper with respect to L as (X, M) is quasi-proper, $\operatorname{pr}_{M^\circ}^* L$ is not $\mathbb{Q}$-linearly equivalent to zero, so $\operatorname{rank}\operatorname{coker}\mathscr{L} = \operatorname{rank}\operatorname{Pic}(X, M^\circ) - 1$.

The class of a torus-invariant prime divisor $\tilde{D}_{\mathbf{m}} \in \operatorname{Div}(X, M)$ is contained in the minimal face of $\operatorname{Eff}^1(X, M)$ containing $a((X, M), L)\operatorname{pr}_M^* L + K_{(X,M)}$ if and only if $\mathbf{m} \in \Gamma_M \setminus \Gamma_{M^\circ}$, by construction of the pair (X, M°). Since the effective cone $\operatorname{Eff}^1(X, M)$ is generated by torus-invariant divisors by Proposition 5.1.2, this implies that $b(\mathbb{Q}, (X, M), L) = \operatorname{rank}\operatorname{Pic}(X, M^\circ)$ finishing the proof. □

Proof of Lemma 5.2.11. Theorem 5.2.12 implies that

$$N_{(X,M),L,S}(B) = B^{a((X,M),L)}(Q(\log B) + O(B^{-\theta})),$$

where Q is a polynomial has degree at most $b(\mathbb{Q}, (X, M), L) - 1$ and $\theta > 0$. This finishes the proof of the lemma. □

5.2.5 Part 2 of the proof of Theorem 5.2.5: computing the leading constant

Now we will show that the polynomial Q has the expected degree for any big and nef $\mathbb{Q}$-divisor L. We will furthermore compute the leading constant under the assumption that the $\mathbb{Q}$-divisor L is toric adjoint rigid with respect to (X, M).

We first notice that it suffices to prove Theorem 5.2.5 for adjoint rigid and toric adjoint rigid $\mathbb{Q}$-divisors L satisfying $a((X, M), L) = 1$.

Proposition 5.2.20. *To show that the polynomial Q obtained in Lemma 5.2.11 has degree $b(\mathbb{Q}, (X, M), L) - 1$, it suffices to assume that $a((X, M), L) = 1$, $S = 1$ and L is toric adjoint rigid with respect to (X, M).*

Proof. We can assume without loss of generality that $a((X, M), L) = 1$ since the height function satisfies $H_{tL}(\mathbf{x}) = H_L(\mathbf{x})^t$ for any $t \in \mathbb{Q}_{>0}$, and thus $N_{(X,M),tL,S}(B) = N_{(X,M),L,S}(B^t)$. The pair (X, M°) is a pair that is quasi-proper with respect to L, and L is toric adjoint rigid with respect to (X, M°). Furthermore $a((X,M),L) = a((X,M^\circ),L)$ and $b(\mathbb{Q},(X,M),L) = b(\mathbb{Q},(X,M^\circ),L) = \operatorname{rank}\operatorname{Pic}(X,M^\circ)$ as in the proof of Proposition 5.2.19. By Lemma 5.2.11, we know that $N_{M,L,S}(B) = B^{a((X,M),L)}(Q(\log B) + O(B^{-\theta}))$ as $B \to \infty$ for some $\theta > 0$ and some polynomial Q of degree at most $b(\mathbb{Q},(X,M),L) - 1$. Note that furthermore

$$N_{(X,M),L,S}(B) \geq N_{(X,M^\circ),L,1}(B).$$

If we assume that Theorem 5.2.5 is true if $a((X,M),L) = 1$, $S = 1$ and L is toric adjoint rigid with respect to (X,M), then this implies

$$N_{(X,M^\circ),L,1}(B) = B^{a((X,M),L)}(Q'(\log B) + O(B^{-\theta})),$$

for some polynomial Q' of degree $b(\mathbb{Q},(X,M),L) - 1$. Now the basic inequality

$$N_{(X,M),L,S}(B) \geq N_{(X,M^\circ),L,1}(B)$$

implies that the degree of Q is at least the degree of Q', and thus Q has degree $b((X,M),L) - 1$ as well. $\square$

Assumption 5.2.21. Henceforth we assume that L is adjoint rigid with respect to (X,M), or that it is toric adjoint rigid with respect to (X,M) and $S = 1$, and in these cases we will compute the leading coefficient of the polynomial Q.

Assumption 5.2.22. The constant C_∞ in Theorem 5.2.5 does not depend on the choice of the integers $d_1, \dots, d_n$ determining the pair $(X, \overline{M^\circ})$ as in Notation 5.1.12. Therefore, we will assume that the integers are chosen large enough to ensure $a(\mathbb{Q},(X,M),L) = a(\mathbb{Q},(X,\overline{M^\circ}),L)$. Note that such integers exist as (X, M°) is quasi-proper with respect to L.

We will prove Theorem 5.2.5 using another theorem of de la Bretèche.

Theorem 5.2.23. *[dlBre01b, Théorème 2(ii), Remarques (ii)] In the setting of Theorem 5.2.12, assume that the following additional conditions are satisfied:*

(C1) There exists a function $\tilde{H}$ such that $H(\mathbf{s}) = \tilde{H}(l_1(\mathbf{s}), \dots, l_{\tilde{n}}(\mathbf{s}))$;

(C2) the vector $(1, \dots, 1) \in \mathbb{R}^k$ is a strictly positive linear combination of $l_1, \dots, l_{\tilde{n}}$;

(C3) $l_1(\boldsymbol{\alpha}) = \dots = l_{\tilde{n}}(\boldsymbol{\alpha}) = 1$.

Then the polynomial Q satisfies the relation

$$Q(\log B) = C_0 B^{-\sum_{j=1}^k \alpha_j} \mathrm{Volume}(D(B)) + O(\log(B)^{\rho-1}),$$

as $B \to \infty$. Here $\rho := \tilde{n} - \operatorname{rank}(l_1, \dots, l_{\tilde{n}})$, $C_0 := H(0, \dots, 0)$ and

$$D(B) = \left\{ \mathbf{y} \in [1,\infty)^{\tilde{n}} \;\middle|\; \prod_{i=1}^{\tilde{n}} y_i^{l_i(\mathbf{e}_j)} \leq B \quad \forall j = 1, \dots, k \right\}.$$

We will apply this theorem for the same series as in the proof of Lemma 5.2.11. We thus only need to verify conditions (C1), (C2) and (C3) and then estimate the volume and show that $H(0, \dots, 0) \neq 0$. Due to the way we chose $\mathscr{L}$, condition (C3) is trivially satisfied. We first show that condition (C2) is satisfied.

Proposition 5.2.24. *If (X, M) is a smooth toric pair such that $0 \in \operatorname{Pic}(X, M)$ is a toric rigid divisor, then the monoid $N_M^+ \subset N$ introduced in Definition 3.2.2 is a lattice.*

Proof. We argue by contradiction, and assume that N_M^+ is not a lattice. Then the cone generated by N_M^+ is not a vector space, and thus there exists a linear form $f\colon N_{\mathbb{R}} \to \mathbb{R}$ such that the half-space $H = \{\mathbf{n} \in N_{\mathbb{R}} \mid f(\mathbf{n}) \geq 0\}$ contains N_M^+ but not $-N_M^+$. Since (X, M) is smooth, N_M^+ is finitely generated, so the linear form f can be chosen such that it restricts to a homomorphism $N \to \mathbb{Z}$, i.e. to an element in $N^\vee$. Using the description of $\operatorname{Pic}(X, M)$ given in Proposition 5.1.3, this implies that the divisor $\sum_{\mathbf{m}\in\Gamma_M} f(\phi(\mathbf{m}))\tilde{D}_{\mathbf{m}}$ is linearly equivalent to 0. By construction $f(\phi(\mathbf{m})) \geq 0$ for all $\mathbf{m} \in \Gamma_M$ and $f(\phi(\mathbf{m})) \neq 0$ for some $\mathbf{m} \in \Gamma_M$, so this is a nontrivial torus-invariant effective divisor. This is in contradiction with the fact that 0 is toric adjoint rigid with respect to (X, M). □

Using the above proposition and the fact that L is toric adjoint rigid with respect to (X, M), there exist coefficients $c_{\mathbf{m}} > 0$ corresponding to the generators $\mathbf{m} \in \Gamma_{M^\circ}$ such that $\sum_{\mathbf{m}\in\Gamma_{M^\circ}} c_{\mathbf{m}}\phi(\mathbf{m}) = 0$, where $\phi\colon \mathbb{N}^n \to N$ is the homomorphism in Definition 3.2.2. Therefore the sum

$$\sum_{\mathbf{m}\in\Gamma_{M^\circ}} l_{\mathbf{m}}(\mathbf{e}_j)c_{\mathbf{m}} = \sum_{\mathbf{m}\in\Gamma_{M^\circ}} (a_{\mathbf{m}} - \langle \mu_L(\sigma_j), \phi(\mathbf{m})\rangle)c_{\mathbf{m}} = \sum_{\mathbf{m}\in\Gamma_{M^\circ}} c_{\mathbf{m}} \sum_{i=1}^{n} a_i m_i > 0$$

does not depend on j, and thus

$$\sum_{\mathbf{m}\in\Gamma_{M^\circ}} l_{\mathbf{m}}c_{\mathbf{m}} = \sum_{\mathbf{m}\in\Gamma_{M^\circ}} c_{\mathbf{m}} \sum_{i=1}^{n} a_i m_i \cdot (1, \ldots, 1),$$

so $\mathcal{B} = (1, \ldots, 1)$ is a positive linear combination of the linear forms in $\mathscr{L}$, and hence condition (C2) of Theorem 5.2.23 is satisfied. Finally, condition (C1) will follow from the following proposition.

Proposition 5.2.25. *The $\mathbb{Q}$-divisor L is adjoint rigid with respect to (X, M) if and only if for every $i = 1, \ldots, n$ the linear form $l^{(i)}$ lies in the linear span of $\mathscr{L}$. Similarly, the divisor L is toric adjoint rigid with respect to (X, M) if and only if for every $\mathbf{m} \in \mathfrak{M}$ the linear form $l_{\mathbf{m}}$ lies in the linear span of $\mathscr{L}$.*

Proof. We give the proof for the toric adjoint rigid case, and we note that the adjoint rigid case is proved analogously. By Corollary 5.2.16, any representative $D = a_1 D_1 + \cdots + a_n D_n$ of $a((X, M), L)L$ with $a_1, \ldots, a_n > 0$ can be written as $D = \sum_{j=1}^{k} \alpha_j L(\sigma_j)$ with $\alpha_1, \ldots, \alpha_k > 0$, for some solution $\boldsymbol{\alpha}$ of the linear program $\mathscr{P}_M$. From this we obtain the expression

$$\operatorname{pr}_M^* D + D_{(X,M)} = \sum_{\mathbf{m}\in\Gamma_M} (l_{\mathbf{m}}(\boldsymbol{\alpha}) - 1)\tilde{D}_{\mathbf{m}},$$

which implies that L is toric adjoint rigid with respect to (X, M) if and only if for every $\mathbf{m} \in \mathfrak{M}$ the value $l_{\mathbf{m}}(\boldsymbol{\alpha})$ does not depend on the choice of a solution $\boldsymbol{\alpha}$ to the linear program $\mathscr{P}_M$.

If the linear form $l_{\mathbf{m}}$ lies in the linear span of $\mathscr{L}$ for every $\mathbf{m} \in \mathfrak{M}$, then the the values of the linear forms in $\mathscr{L}$ evaluated at $\boldsymbol{\alpha}$ determine the value of $l_{\mathbf{m}}(\boldsymbol{\alpha})$ for all $\mathbf{m} \in \mathfrak{M}$. Since for every $l \in \mathscr{L}$ we have $l(\boldsymbol{\alpha}) = 1$ by definition, we therefore see that L is toric adjoint rigid.

Conversely, assume that $l_{\mathbf{m}'}$ is a linear form not in the span of $\mathscr{L}$ for some $\mathbf{m}' \in \mathfrak{M}$. Then there exists $\boldsymbol{\beta} \in \mathbb{R}^k$ such that $l_{\mathbf{m}'}(\boldsymbol{\beta}) = 1$ but $l(\boldsymbol{\beta}) = 0$ for all $l \in \mathscr{L}$. Let D be a representative of L satisfying Assumption 5.2.17, and take $\boldsymbol{\alpha}$ such that $D = \sum_{j=1}^{k} \alpha_j L(\sigma_j)$ and $\alpha_j > 0$ for all $j = 1, \ldots, k$. For every $\mathbf{m} \in \Gamma_M$ such that $l_{\mathbf{m}} \notin \mathscr{L}$ we have $l_{\mathbf{m}}(\boldsymbol{\alpha}) > 1$, by construction of $\mathscr{L}$. Therefore, there exists $\epsilon > 0$ such that $l_{\mathbf{m}}(\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta}) \geq 1$ for all $\mathbf{m} \in \mathfrak{M}$ and $\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta} > 0$. This implies that $\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta} \in \mathbb{R}^k_{>0}$ is also a solution to $\mathscr{P}_M$. Since $l_{\mathbf{m}'}(\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta}) \neq l_{\mathbf{m}'}(\boldsymbol{\alpha})$, this implies that L is not toric adjoint rigid. □

Note that $F(\mathbf{s})$ is always a function of the linear forms $l^{(1)}(\mathbf{s}), \ldots, l^{(n)}(\mathbf{s})$. Furthermore if $S = 1$, then $F(\mathbf{s})$ is a function of the linear forms $l_{\mathbf{m}}(\mathbf{s})$ for $\mathbf{m} \in \Gamma_M$. Hence condition (C1) is satisfied by Proposition 5.2.25, since either L is adjoint rigid or L is toric adjoint rigid and $S = 1$. Thus we can apply Theorem 5.2.23 to determine the leading constant. By this theorem, the polynomial Q giving the asymptotic satisfies

$$Q(\log B) = 2^{\dim X} C_0 I(B)/B^{a((X,M),L)} + O((\log B)^{b(\mathbb{Q},(X,M),L)-2}),$$

where

$$C_0 = H(0, \ldots, 0)$$

and $I(B)$ is the volume of the domain

$$D(B) = \left\{ \mathbf{x} \in [1,\infty)^{\Gamma_{M^\circ}} \,\middle|\, \prod_{\mathbf{m} \in \Gamma_{M^\circ}} x_{\mathbf{m}}^{l_{\mathbf{m}}(\mathbf{e}_j)} \leq B, \forall j = 1, \ldots, k \right\}.$$

Thus to prove Theorem 5.2.5, it remains to compute C_0 and estimate $I(B)$ as $B \to \infty$. First we will compute C_0.

Proposition 5.2.26. *The value of H at the origin is equal to the infinite product*

$$C_0 = \prod_{\substack{p \text{ prime} \\ p \mid S}} (1 - p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m} \in \mathfrak{M}_{\mathrm{red}}} p^{-a_{\mathbf{m}}} \times \prod_{\substack{p \text{ prime} \\ p \nmid S}} (1 - p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m} \in \mathbb{N}^n_{\mathrm{red}}} p^{-a_{\mathbf{m}}}$$

and this quantity is positive. Here we recall $a_{\mathbf{m}} = \sum_{i=1}^{n} m_i a_i$. Furthermore, if L is adjoint rigid, then $\#\Gamma_{M^\circ} = \dim(X) + b(\mathbb{Q}, (X, M), L)$.

Proof. Note that

$$H(\mathbf{s}) = G(\mathbf{s} + \boldsymbol{\alpha}) \prod_{\mathbf{m} \in \Gamma_{M^\circ}} l_{\mathbf{m}}(\mathbf{s}) \zeta(l_{\mathbf{m}}(\mathbf{s}) + 1),$$

and

$$G_p(\mathbf{s}) = \left(\sum_{\mathbf{m}\in\mathfrak{M}} p^{-l_{\mathbf{m}}(\mathbf{s})}\right) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} (1 - p^{-l_{\mathbf{m}}(\mathbf{s})})$$

for any prime number p not dividing S and similarly

$$G_p(\mathbf{s}) = \left(\sum_{\mathbf{m}\in\mathbb{N}^n_{\mathrm{red}}} p^{-l_{\mathbf{m}}(\mathbf{s})}\right) \prod_{\mathbf{m}\in\Gamma_{M^\circ}} (1 - p^{-l_{\mathbf{m}}(\mathbf{s})})$$

for any prime number p dividing S.

Thus the limit $\lim_{z\to 0} z\zeta(z+1) = 1$ implies $H(0,\dots,0) = G(\boldsymbol{\alpha})$. This gives the desired identity for C_0 under the assumption that the product converges. Each term C_p in the product is positive and $C_p = 1 + O(p^{-c})$, where the implied constant is independent of p and c is the smallest between 2 and the minimum of all values $l_{\mathbf{m}}(\boldsymbol{\alpha}) = a_{\mathbf{m}}$ for $\mathbf{m} \in \mathfrak{M} \setminus \mathfrak{M}^\circ$, so the product thus converges to a positive constant.

Now assume that L is adjoint rigid with respect to (X, M). We claim that this implies that the map $N^\vee \to \mathrm{Div}_T(X, M)$ is injective. If it were not injective, then there exists $\mu \in N^\vee$ such that $\mathrm{pr}_M^* \operatorname{div}(\chi^\mu) = 0$. But then there exists $c \in \mathbb{Q}$ such that $\mathrm{pr}_M^* \operatorname{div}(c + \chi^\mu)$ is a nontrivial effective divisor on (X, M). This implies that $0 \in \mathrm{Div}(X, M)$ is not a rigid divisor, which contradicts the fact that L is adjoint rigid with respect to (X, M). Thus Proposition 5.1.3 implies $\#\Gamma_{M^\circ} = \dim(X) + \operatorname{rank} \operatorname{Pic}(X, M^\circ)$. Furthermore, the proof of Proposition 5.2.19 shows $\operatorname{rank} \operatorname{Pic}(X, M^\circ) = b(\mathbb{Q}, (X, M), L)$, so we obtain the desired expression for $\#\Gamma_{M^\circ}$. □

Now it remains to estimate the volume of the set $D(B)$. In order to simplify notation, we assume $M = M^\circ$, which we can do without loss of generality as the definition of $D(B)$ only depends on (X, M°) and not on (X, M). The set $D(B)$ is a generalization of the set $D(B)$ defined by Salberger [Sal98, Notation 11.28] in his study of rational points on split toric varieties, and we will use the same approach he used to estimate its volume.

We regard $D(B)$ as a closed subset of the real locus of the universal torsor of (X, M), where the universal torsor is as in Definition 5.1.15. As we assumed $M = M^\circ$ and $a((X, M), L) = 1$, we have $\mathrm{pr}_M^* D = -D_{(X,M)}$. In order to estimate the volume of $D(B)$, Salberger splits it up as $D(B) = \cup_{\sigma\in\Sigma_{\max}} D(B, \sigma)$ using what he calls the toric canonical splitting [Sal98, Notation 11.31]. We will similarly split up $D(B)$, but we will use the splitting induced by the fan $\Sigma_{\overline{M}}$ as given in Notation 5.1.12, rather than Σ. The fan $\Sigma_{\overline{M}}$ has the property that for all $\mathbf{m} \in \Gamma_M$, the ray spanned by an element $\phi(\mathbf{m}) \in N$ lies in $\Sigma_{\overline{M}}$, and all rays in $\Sigma_{\overline{M}}$ are of this form. This property will aid in computing $D(B)$.

Since the dense torus in X is $U = \mathrm{Hom}(N^\vee, \mathbb{G}_m)$, the real locus of the torus is $U(\mathbb{R}) = \mathrm{Hom}(N^\vee, \mathbb{R}^\times)$. By composing with the logarithm of the absolute value $\mathbb{R}^\times \to \mathbb{R}\colon x \mapsto \log|x|$, we obtain a homomorphism

$$U(\mathbb{R}) \to \mathrm{Hom}(N^\vee, \mathbb{R}^\times) = N_{\mathbb{R}}.$$

Recall that U_M is the dense torus in the universal torsor Y_M of (X, M). The morphism $Y_M \to X$ induces a homomorphism $U_M(\mathbb{R}) \to U(\mathbb{R})$. By composing these homomorphism with each other, we obtain a homomorphism

$$\operatorname{Log}_M \colon U_M(\mathbb{R}) \to \operatorname{Hom}(N^\vee, \mathbb{R}^\times) = N_\mathbb{R}.$$

For $\sigma \in \Sigma_{\overline{M},\max}$, write $C_{M,\sigma,0}(\mathbb{R})$ for the inverse image of $-\sigma$ under the map Log_M and write $D(B,\sigma) = D(B) \cap C_{M,\sigma,0}(\mathbb{R})$. Since for any two distinct maximal cones σ, σ', their intersection $\sigma \cap \sigma'$ lies in a proper subspace of $N_\mathbb{R}$, the intersection of $D(B,\sigma) \cap D(B,\sigma')$ has Lebesgue measure zero and thus

$$I(B) = \int_{D(B)} \mathrm{d}\mathbf{x} = \sum_{\sigma \in \Sigma_{\overline{M},\max}} \int_{D(B,\sigma)} \mathrm{d}\mathbf{x}, \tag{5.2.2}$$

where the measure is the standard Lebesgue measure on $\mathbb{R}^{\Gamma_M}$. We will compute $\int_{D(B,\sigma)} \mathrm{d}\mathbf{x}$ for each maximal cone $\sigma \in \Sigma_{\overline{M}}$.

As in [Sal98, Proposition 11.22], we can describe when a point lies in $C_{M,\sigma,0}(\mathbb{R})$. The cone σ contains exactly $d = \dim X$ rays. Let d_1 be the number of rays in σ which lie in Σ_M. Let $r = \#\Sigma_{M(\sigma)}(1) - d$ be the number of raysin $\Sigma_{M(\sigma)}$ which lie outside of σ. Order the rays $\rho_1, \dots, \rho_{r+d}$ in $\Sigma_{M(\sigma)}$ such that $\rho_{r+1}, \dots, \rho_{r+d_1}$ are the rays in both σ and Σ_M and $\rho_{r+1}, \dots, \rho_{r+d}$ are the rays in σ. Let $\mathbf{m}_1, \dots, \mathbf{m}_{r+d} \in \Gamma_{M(\sigma)}$ be the elements corresponding to the rays $\rho_1, \dots \rho_{r+d}$, and set $n^{(i)} = \phi(\mathbf{m}_{r+i})$ for $i = 1, \dots, d$. As $n^{(1)}, \dots, n^{(d)}$ are integer multiples of the ray generators of $\rho_{r+1}, \dots, \rho_{r+d}$, they freely generate a finite-index sublattice N_σ of N. Let $(\mu^{(1)}, \dots, \mu^{(d)})$ be the corresponding dual $\mathbb{Z}$-basis of $N_\sigma^\vee \supset N^\vee$ and set

$$D(i) = \sum_{\mathbf{m} \in \Gamma_M} \langle \mu^{(i)}, \phi(\mathbf{m}) \rangle \tilde{D}_\mathbf{m} \in \operatorname{Div}_T(X, M)_\mathbb{Q},$$

where $\tilde{D}_\rho$ is the prime divisor on (X, M) corresponding to the ray ρ. Note that

$$D(i) = \operatorname{pr}_M^* \sum_{\rho \in \Sigma(1)} \langle \mu^{(i)}, n_\rho \rangle D_\rho$$

by construction, so $D(i)$ is $\mathbb{Q}$-linearly equivalent to 0. Furthermore, since $\mu^{(1)}, \dots, \mu^{(d)}$ is a basis for $N_\mathbb{Q}^\vee$, $(D(1), \dots, D(n))$ is a basis for the vector space of all torus-invariant $\mathbb{Q}$-divisors on (X, M) linearly equivalent to 0.

Proposition 5.2.27. *Let $\mathbf{x} \in [1, \infty)^{\Gamma_M}$. Then $\mathbf{x} \in C_{M,\sigma,0}(\mathbb{R})$ if and only if $\mathbf{x}^{D(i)} \leq 1$ for all $i = 1, \dots, d$.*

Proof. The proof is identical to the proof of [Sal98, Proposition 11.22]. □

For $i \in \{1, \dots, r + d_1\}$, we write $\tilde{D}_i := \tilde{D}_{\mathbf{m}_i} \in \operatorname{Div}(X, M)$ for the divisor corresponding to the ray $\rho_i \subset \Sigma_M$.

Set

$$E(i) := \tilde{D}_{r+i} - D(i) \text{ if } \rho_{r+i} \in \Sigma_M(1)$$

and

$$E(i) := -D(i) \text{ if } \rho_{r+i} \in \Sigma_{\overline{M}}(1) \setminus \Sigma_M(1).$$

By construction, the divisor $E(i)$ is supported on the divisors $\tilde{D}_\mathbf{m} \in \operatorname{Div}(X, M)$ such that $\phi(\mathbf{m}) \notin \sigma$ and it is $\mathbb{Q}$-linearly equivalent to $\tilde{D}_{r+i}$ if $\rho_{r+i} \in \Sigma_M(1)$ and otherwise it is $\mathbb{Q}$-linearly equivalent to 0.

Notation 5.2.28. Note that for a maximal cone $\sigma \in \Sigma_{\overline{M}}$ and a $\mathbb{Q}$-divisor class $L' \in \operatorname{Pic}(X, \overline{M})_{\mathbb{Q}}$, there is a unique representative $L'(\sigma) \in \operatorname{Div}_T(X, M)_{\mathbb{Q}}$ of L' supported only on the divisors $\tilde{D}_{\mathbf{m}} \in \operatorname{Div}(X, \overline{M})$ with $\phi(\mathbf{m}) \notin \sigma$. For a maximal cone $\sigma \in \Sigma_{\overline{M}}$, we write $(\operatorname{pr}_M^* L)(\sigma)$ for the restriction of $(\operatorname{pr}_{\overline{M}}^* L)(\sigma)$ to (X, M). Similarly, we write $D_{(X,M)}(\sigma) \in \operatorname{Div}(X, M)_{\mathbb{Q}}$ by viewing $D' = D_{(X,M)}$ as a divisor on $(X, \overline{M})$ and by restricting $D'(\sigma)$ to (X, M).

In particular, for a maximal cone $\sigma \in \Sigma_{\overline{M}}$ and $L \in \operatorname{Pic}(X)_{\mathbb{Q}}$,

$$(\operatorname{pr}_M^* L)(\sigma) = \operatorname{pr}_M^*(L(\overline{\sigma})),$$

where $\overline{\sigma}$ is the unique maximal cone in Σ containing σ.

Lemma 5.2.29. *$D(B, \sigma)$ is the set of all $(x_1, \ldots, x_{r+d_1}) \in X_{M,0}(\mathbb{R}) \subset \mathbb{R}^{r+d_1}$ satisfying*

1. $\min(x_1, \ldots, x_{r+d_1}) \geq 1$,
2. $\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma)} \leq B$,
3. $\mathbf{x}^{E(i)} \geq x_{r+i}$, *for all* $i = 1, \ldots, d_1$, *and* $\mathbf{x}^{E(i)} \geq 1$, *for all* $i = d_1 + 1, \ldots, d$.

Proof. By Proposition 5.2.27, $\mathbf{x} \in [1, \infty)^{\Gamma_M}$ lies in $\mathbf{x} \in C_{M,\sigma,0}(\mathbb{R})$ if and only if the first and third conditions are satisfied. Since $\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma)} = \prod_{\mathbf{m} \in \Gamma_M} x_{\mathbf{m}}^{l_{\mathbf{m}}(\mathbf{e}_j)}$ for the unique maximal cone $\sigma_j \in \Sigma$ containing σ, it remains to show that $\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma)} \leq B$ is equivalent to $\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma')} \leq B$ for all $\sigma' \in \Sigma_{\overline{M}}$. Because the divisors $D(1), \ldots, D(d)$ generate the kernel of $\operatorname{Div}_T(X, M)_{\mathbb{Q}} \to \operatorname{Pic}(X, M)_{\mathbb{Q}}$, we must have $(\operatorname{pr}_M^* L)(\sigma') = (\operatorname{pr}_M^* L)(\sigma) + \sum_{i=1}^{d} c_i D(i)$, for $c_1, \ldots, c_d \in \mathbb{Q}$. Let $\overline{\sigma}$ and $\overline{\sigma'}$ be the unique maximal cones in Σ containing σ and σ', respectively. By considering the pullbacks of $L(\overline{\sigma})$ and $L(\overline{\sigma'})$ to (X, M), c_i is equal to the coefficient of $\tilde{D}_{r+i} \in \operatorname{Div}(X, M)$ in the pullback of $L(\overline{\sigma'}) - L(\overline{\sigma})$ to (X, M), for all $i \in \{1, \ldots, d\}$. As the coefficient of $\tilde{D}_{r+i} \in \operatorname{Div}(X, M)$ in $\operatorname{pr}_M^* L(\overline{\sigma})$ is zero, and L is nef, me must have $c_i \geq 0$ for all $i \in \{1, \ldots, d\}$. Therefore Proposition 5.2.27 implies

$$\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma')} \leq \mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma)},$$

as desired. □

Similarly, we define $\Omega(B, \sigma)$ to be the set of all $(x_1, \ldots, x_r)$ satisfying

1. $\min(x_1, \ldots, x_r) \geq 1$,
2. $\mathbf{x}^{(\operatorname{pr}_M^* L)(\sigma)} \leq B$,
3. $\mathbf{x}^{E(i)} \geq 1$, for all $i = 1, \ldots, d$.

By Fubini's theorem, we can compute $D(B, \sigma)$ by first integrating with respect to $(x_{r+1}, \ldots, x_{r+d_1})$ and then with respect to $(x_1, \ldots, x_r)$:

$$\int_{D(B,\sigma)} \mathrm{d}\mathbf{x} = \int_{\Omega(B,\sigma)} \prod_{i=1}^{d_1} (\mathbf{x}^{E(i)} - 1) \, \mathrm{d}x_1 \ldots \mathrm{d}x_r \tag{5.2.3}$$

This equality combined with

$$\sum_{i=1}^{d_1} E(i) = -D_{(X,M)}(\sigma) - \sum_{i=1}^{r} \tilde{D}_i \tag{5.2.4}$$

implies

$$\int_{D(B,\sigma)} \mathrm{d}\mathbf{x} = \int_{\Omega(B,\sigma)} \mathbf{x}^{-D_{(X,M)}(\sigma)} \prod_{i=1}^{d_1} (1 - \mathbf{x}^{-E(i)}) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}. \tag{5.2.5}$$

Let $T_{M(\sigma)} \subset Y_{M(\sigma)}$ be the Picard torus for the pair $(X, M(\sigma))$ as in Definition 5.1.15. The projection $Y_{M(\sigma)} \subset \mathbb{A}^{r+d}_{\mathbb{Q}} \to \mathbb{A}^r_{\mathbb{Q}}$ onto the first r coordinates induces an analytic homomorphism

$$T_{M(\sigma)}(\mathbb{R}) \to (\mathbb{R}^\times)^r$$
$$(x_1, \ldots, x_{r+d}) \mapsto (x_1, \ldots, x_r)$$

of Lie groups. The identity component $T_{M(\sigma)}(\mathbb{R})^+$ of $T_{M(\sigma)}(\mathbb{R}) \subset \mathbb{R}^n$ is the set of points with positive coordinates, and the analytic homomorphism restricts to an analytic isomorphism $T_{M(\sigma)}(\mathbb{R})^+ \to \mathbb{R}^r_{>0}$. For $i \in \{1, \ldots, d_1\}$, the image of $E(i)$ in $\mathrm{Div}_T(X, M(\sigma))$ is $\mathbb{Q}$-linearly equivalent to $\tilde{D}_{r+i}$, so $\mathbf{x}^{E(i)} = x_{r+i}$ for all $\mathbf{x} \in T_{M(\sigma)}(\mathbb{R})^+$. Furthermore, since $-D_{(X,M)}(\sigma)$ is $\mathbb{Q}$-linearly equivalent to $-D_{(X,M)}$, viewed as $\mathbb{Q}$-divisors on $(X, M(\sigma))$, the isomorphism identifies the set $\Omega(B, \sigma)$ with the subset $F(B) \subset T_{M(\sigma)}(\mathbb{R})$ given by the elements $(x_1, \ldots, x_{r+d})$ with

1. $\min(x_1, \ldots, x_r) \geq 1$,
2. $\mathbf{x}^{(\mathrm{pr}_M^* L)(\sigma)} \leq B$.

Under the isomorphism $T_{M(\sigma)}(\mathbb{R})^+ \to \mathbb{R}^r_{>0}$, the differential form $\frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}$ on $(\mathbb{R}^\times)^r$ corresponds to the torus-invariant differential form $\frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}$ on $T_{M(\sigma)}(\mathbb{R})^+$.

Consequently, we find the following analogue of [Sal98, Equation (11.37)]

$$\int_{D(B,\sigma)} \mathrm{d}\mathbf{x} = \int_{F(B)} \mathbf{x}^{-D_{(X,M)}} \prod_{i=1}^{d_1} (1 - 1/x_{r+i}) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}. \tag{5.2.6}$$

We will now first focus on estimating

$$I(B, \sigma) = \int_{F(B)} \mathbf{x}^{-D_{(X,M)}} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}, \tag{5.2.7}$$

and then we show in Lemma 5.2.31 that $\int_{D(B,\sigma)} \mathrm{d}\mathbf{x} \sim I(B, \sigma)$ as $B \to \infty$.

There is an analytic isomorphism

$$\psi \colon T_{M(\sigma)}(\mathbb{R})^+ \to V := \mathrm{Hom}(\mathrm{Pic}(X, M(\sigma)), \mathbb{R})$$

given by $(x_1, \ldots, x_{r+d}) \mapsto (y_1, \ldots, y_{r+d})$, where $y_i = \log x_i$ for $i \in \{1, \ldots, r+d\}$.

Let $\mathrm{Hom}_{\geq 0}(\mathrm{Pic}(X, M(\sigma)), \mathbb{R}) \subset V$ be set of linear functions which are nonnegative on effective divisor classes on $(X, M(\sigma))$. Then the isomorphism ψ sends the set $T^{\geq 1}_{M(\sigma)}(\mathbb{R})$ consisting of all $(x_1, \dots, x_{r+d}) \in T_{M(\sigma)}(\mathbb{R})^+$ with $x_1, \dots, x_{r+d} \geq 1$ to $\operatorname{Hom}_{\geq 0}(\operatorname{Pic}(X, M(\sigma)), \mathbb{R})$.

Let $b = \log B$. The image $E_b := \psi(F(B))$ is the set of all $\varphi \in \operatorname{Hom}_{\geq 0}(\operatorname{Pic}(X, M(\sigma)), \mathbb{R})$ with $\varphi(\operatorname{pr}_M^* L) \leq b$. Let ν be the Haar measure on V such that the volume of V/Λ is 1 for the lattice $\Lambda := \operatorname{Hom}(\operatorname{Pic}(X, M(\sigma)), \mathbb{Z})$. Under the analytic isomorphism ψ, the differential form $\frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_r}{x_r}$ gets sent to $\mathrm{d}y_1 \dots \mathrm{d}y_r$. Recall that $I(\sigma)$ is the index of $\langle \tilde{D}_1, \dots, \tilde{D}_r \rangle$ inside $\operatorname{Pic}(X, M(\sigma))$. As $\langle \tilde{D}_1, \dots, \tilde{D}_r \rangle$ is torsion-free, it has index $\frac{I(\sigma)}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}}$ in $\operatorname{Pic}(X, M(\sigma))/\{\text{torsion}\}$. Thus the lattice Λ has index $\frac{I(\sigma)}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}}$ in $\langle [\tilde{D}_1]^*, \dots, [\tilde{D}_r]^* \rangle$. This implies

$$\mathrm{d}y_1 \dots \mathrm{d}y_r = \frac{I(\sigma)}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}} \mathrm{d}\nu,$$

and thus Equation (5.2.7) becomes

$$I(B, \sigma) = \frac{I(\sigma)}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}} \int_{E_b} \exp(y_1 + \dots + y_{r+d_1}) \,\mathrm{d}\nu. \tag{5.2.8}$$

We write $\tilde{a}_i$ for the coefficient of $\tilde{D}_i$ in $\operatorname{pr}_{M(\sigma)}^* L$. Note that $\operatorname{pr}_{M(\sigma)}^* L + D_{(X,M)} = \sum_{i=r+d_1+1}^{r+d} \tilde{a}_i \tilde{D}_i$, and $\tilde{a}_i = 1$ if $i \leq r + d_1$.

Let $V'' \subset \mathbb{R}^{d-d_1}$ be the vector space of linear functions $\langle [\tilde{D}_{r+d_1+1}], \dots, [\tilde{D}_{r+d}] \rangle \to \mathbb{R}$. The projection $V \to V''$ given by $(y_1, \dots, y_{r+d}) \mapsto (y_{r+d_1+1}, \dots, y_{r+d})$ implies the existence of a splitting $V \cong V' \times V''$, where $V' = \{(y_1, \dots, y_{r+d}) \in V \mid y_{r+d_1+1}, \dots, y_{r+d} = 0\}$. The space V' is naturally identified with $\operatorname{Hom}(\operatorname{Pic}(X, M), \mathbb{R})$, and the Haar measure ν' on V' induced by ν on V is the measure such that the volume of V'/Λ' is 1, where $\Lambda' := \operatorname{Hom}(\operatorname{Pic}(X, M), \mathbb{Z})$. Similarly ν induces the Haar measure ν'' on V'' such that V''/Λ'' has volume equal to 1, where Λ'' is the image of Λ in V''. Under the isomorphism $V \cong V' \times V''$, the measure ν corresponds to the product measure $\nu' \times \nu''$, so Fubini's theorem implies that

$$\begin{aligned} I(B, \sigma) =& \frac{I(\sigma)}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}} \\ &\times \int_{E_b \cap V'} \exp(y_1 + \dots + y_{r+d_1}) \operatorname{Volume}(Z_\sigma(b - y_1 - \dots - y_{r+d_1})) \,\mathrm{d}\nu' \\ =& \frac{I(\sigma) \operatorname{Volume}(Z_\sigma(1))}{\# \operatorname{Pic}(X, M(\sigma))_{\text{torsion}}} \\ &\times \int_{E_b \cap V'} \exp(y_1 + \dots + y_{r+d_1}) (b - y_1 - \dots - y_{r+d_1})^{\dim(Z_\sigma(1))} \,\mathrm{d}\nu', \end{aligned}$$

where $Z_\sigma(c) = \{(y_{r+d_1+1}, \dots, y_{r+d}) \in V'' \cap [0, \infty)^{d-d_1} \mid \sum_{i=r+d_1+1}^{r+d} \tilde{a}_i y_i \leq c\}$ is a polytope and the volume is with respect to the measure ν''. Note that the polytope $Z_\sigma(1)$ is the polytope Z_σ in Theorem 5.2.5.

Under the identification of V' with $\operatorname{Hom}(\operatorname{Pic}(X, M), \mathbb{R})$, the subset $E_b \cap V'$ is the set of linear forms φ which are nonnegative on effective divisors and such that $\phi(-K_{(X,M)}) \leq b$. Let $\lambda \colon V' \to \mathbb{R}$ be the linear form

$$\lambda(y_1, \dots, y_{r+d_1}) = y_1 + \dots + y_{r+d_1}$$

obtained by evaluating at the anticanonical class $-K_{(X,M)}$ of (X,M). As the volume of the fibre above any $y \in [0,\infty)$ is equal to

$$\#\operatorname{Pic}(X,M)_{\mathrm{torsion}}\alpha_{\mathrm{Peyre}}((X,M),L)y^{b(\mathbb{Q},(X,M),L)-1},$$

where $\alpha_{\mathrm{Peyre}}((X,M),L)$ is as in Remark 5.2.4, integrating along the fibres of λ gives

$$\begin{aligned}\int_{E_b\cap V'} &\exp(y_1+\cdots+y_{r+d_1})(b-y_1-\cdots-y_{r+d_1})^{d-d_1}\,\mathrm{d}\nu' \\ =&\#\operatorname{Pic}(X,M)_{\mathrm{torsion}}\alpha_{\mathrm{Peyre}}((X,M),L) \\ &\times\int_0^b \exp(y)y^{b(\mathbb{Q},(X,M),L)-1}(b-y)^{\dim(Z(1))}\,\mathrm{d}y.\end{aligned}$$

Thus we have shown the equality

$$\begin{aligned}I(B,\sigma)=&\frac{\#\operatorname{Pic}(X,M)_{\mathrm{torsion}}I(\sigma)\alpha_{\mathrm{Peyre}}((X,M),L)\operatorname{Volume}(Z(1))}{\#\operatorname{Pic}(X,M(\sigma))_{\mathrm{torsion}}} \\ &\times\int_0^b \exp(y)y^{b(\mathbb{Q},(X,M),L)-1}(b-y)^{\dim(Z_\sigma)}\,\mathrm{d}y.\end{aligned} \tag{5.2.9}$$

We will now determine the main term in the integral.

Proposition 5.2.30. *For all $r,s\in\mathbb{N}$ and $b\in(1,\infty)$,*

$$\int_0^b \exp(y)y^r(b-y)^s\,\mathrm{d}y = s!\exp(b)b^r + O(\exp(b)b^{r-1})$$

as $b\to\infty$.

Proof. Let

$$I(r,s,b) := \int_0^b \exp(y)y^r(b-y)^s\,\mathrm{d}y.$$

If either r or s is zero, the integral is given by

$$I(r,0,b)=\exp(b)\sum_{k=0}^{r}(-1)^k\frac{r!}{(r-k)!}b^{r-k}+(-1)^{r+1}r!,$$

or

$$I(0,s,b)=\exp(b)s!-\sum_{i=0}^{s}\frac{s!}{(s-i)!}b^{s-i}.$$

Further integrating by parts, we obtain $I(r,s,b)=sI(r,s-1,b)-rI(r-1,s,b)$ if $r,s\geq 1$, which directly implies the result. □

Therefore we find

$$\begin{aligned}I(B,\sigma)=&\frac{\#\operatorname{Pic}(X,M)_{\mathrm{torsion}}I(\sigma)\alpha_{\mathrm{Peyre}}((X,M),L)\operatorname{Volume}(Z_\sigma)\dim(Z_\sigma)!}{\#\operatorname{Pic}(X,M(\sigma))_{\mathrm{torsion}}} \\ &\times B(\log B)^{b(\mathbb{Q},(X,M),L)-1}+O(B(\log B)^{b(\mathbb{Q},(X,M),L)-2}).\end{aligned} \tag{5.2.10}$$

To finish the proof of Theorem 5.2.5 for toric adjoint rigid divisors, all that remains is to show that $\int_{D(B,\sigma)}\mathrm{d}\mathbf{x}\sim I(B,\sigma)$.

Lemma 5.2.31. *For every maximal cone $\sigma \in \Sigma_{\overline{M}}$,*

$$\int_{D(B,\sigma)} \mathrm{d}\mathbf{x} = I(B,\sigma) + O(B(\log B)^{b(\mathbb{Q},(X,M),L)-2})$$

as $B \to \infty$, so (5.2.2) *implies*

$$I(B) = \sum_{\sigma \in \Sigma_{\overline{M}}} I(B,\sigma) + O(B(\log B)^{b(\mathbb{Q},(X,M),L)-2})$$

as $B \to \infty$.

Proof. By (5.2.6), it suffices to show

$$R_i = \int_{F(B)} \mathbf{x}^{-D_{(X,M)}}/x_i \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r} = O(B(\log B)^{b(\mathbb{Q},(X,M),L)-2})$$

as $B \to \infty$, for any $i = r+1, \dots, r+d_1$.

Let (X, M') be the pair given by $\mathfrak{M}' = \mathfrak{M}^\circ \setminus \{\mathbf{m}\}$, where $\mathbf{m}$ is the element corresponding to the coordinate x_i. Then $R_i = \int_{F(B)} \mathbf{x}^{-D_{(X,M')}} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}$. We can estimate this integral in exactly the same way as in the computation of the asymptotic growth of $I(B,\sigma)$, to get

$$R_i = O(B(\log B)^{b(\mathbb{Q},(X,M'),L)-1})$$

as $B \to \infty$. Now since $[\tilde{D}_i]$ does not lie on the minimal face of $\mathrm{Eff}^1(X,M)$ containing $\mathrm{pr}_M^* L + K_{(X,M)}$, we have $b(\mathbb{Q},(X,M'),L) = b(\mathbb{Q},(X,M),L) - 1$, which gives the result. ☐

Putting everything together, we conclude that the polynomial Q satisfies

$$Q(\log B) = 2^{\dim X} C_0 \tilde{C} (\log B)^{b(\mathbb{Q},(X,M),L)-1} + O((\log B)^{b(\mathbb{Q},(X,M),L)-2}),$$

where

$$C_0 = \prod_{\substack{p \text{ prime} \\ p \mid S}} (1-p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m} \in \mathfrak{M}_{\mathrm{red}}} p^{-a_{\mathbf{m}}} \times \prod_{\substack{p \text{ prime} \\ p \nmid S}} (1-p^{-1})^{\#\Gamma_{M^\circ}} \sum_{\mathbf{m} \in \mathbb{N}^n_{\mathrm{red}}} p^{-a_{\mathbf{m}}},$$

and

$$\tilde{C} = \alpha_{\mathrm{Peyre}}((X,M),L) \sum_{\sigma \in \Sigma_{\overline{M},\max}} I(\sigma) C_\infty(\sigma),$$

where

$$C_\infty(\sigma) = \frac{\mathrm{Pic}(X,M)_{\mathrm{torsion}}}{\mathrm{Pic}(X,M(\sigma))_{\mathrm{torsion}}} \mathrm{Volume}(Z_\sigma) \dim(Z_\sigma)!.$$

This finishes the proof of the toric adjoint rigid case of the theorem.

The leading constant in the adjoint rigid case. From now on we will assume that L is adjoint rigid with respect to (X, M). As this implies that L is toric adjoint rigid with respect to (X, M), we can use the expression we derived for the leading coefficient of Q. We have already seen in Proposition 5.2.26 that $\#\Gamma_{M^\circ} = \dim(X) + b(\mathbb{Q}, (X, M), L)$. Therefore, all that remains is to prove

$$\tilde{C} = \alpha_{\mathrm{Peyre}}((X, M), L)C_\infty,$$

where C_∞ is the constant given in Theorem 5.2.5 in the adjoint rigid case. We start by computing the volume of the simplex Z_σ.

Proposition 5.2.32. *The simplex Z_σ has dimension $d - d_1$ and its volume is given by*

$$\mathrm{Volume}(Z_\sigma) = \frac{\mathrm{Pic}(X, M(\sigma))_{\mathrm{torsion}}}{(d-d_1)!\# \mathrm{Pic}(X, M)_{\mathrm{torsion}} \prod_{i=r+d_1+1}^{r+d} \tilde{a}_i},$$

where we recall that $\tilde{a}_i$ is the coefficient of $\tilde{D}_i$ in $\mathrm{pr}_{M(\sigma)}^* L$.

Proof. By Proposition 5.2.25, the $\mathbb{Q}$-divisor class $\mathrm{pr}_{\overline{M}}^* L + K_{(X,\overline{M})}$ is rigid, so the subgroup G of $\mathrm{Pic}(X, M(\sigma))$ generated by the divisors classes $[\tilde{D}_{r+d_1+1}], \dots, [\tilde{D}_{r+d}]$ is a free abelian group of rank $d - d_1$. Therefore $V'' \cong \mathbb{R}^{d-d_1}$, and $Z_\sigma \subset V''$ is a simplex with vertices

$$(\tilde{a}_{r+d_1+1}^{-1}, 0, \dots, 0), \dots, (0, \dots, 0, \tilde{a}_{r+d}^{-1}).$$

Thus the volume of Z_σ is given by

$$\mathrm{Volume}(Z_\sigma) = \frac{\mathrm{Volume}([0,1]^{d-d_1})}{(d-d_1)! \prod_{i=r+d_1+1}^{r+d} \tilde{a}_i}.$$

The volume of the hypercube $[0,1]^{d-d_1}$ with respect to the measure ν'' is the reciprocal of the order of the kernel of the quotient homomorphism $\mathrm{Pic}(X, M(\sigma))/\{\text{torsion}\} \to \mathrm{Pic}(X, M)/\{\text{torsion}\}$ induced by the restriction $\mathrm{Div}(X, M(\sigma)) \to \mathrm{Div}(X, M)$. Since the kernel G of $\mathrm{Pic}(X, M(\sigma)) \to \mathrm{Pic}(X, M)$ is torsion-free, the volume of the hypercube is $\mathrm{Volume}([0,1]^{d-d_1}) = \frac{\# \mathrm{Pic}(X, M(\sigma))_{\mathrm{torsion}}}{\# \mathrm{Pic}(X, M)_{\mathrm{torsion}}}$. □

The previous proposition implies

$$C_\infty(\sigma) = \prod_{\substack{\mathbf{m} \in \Gamma_{\overline{M}} \\ \rho_{\mathbf{m}} \in \sigma}} \frac{1}{a_{\mathbf{m}}}$$

so

$$\tilde{C} = \alpha_{\mathrm{Peyre}}((X, M), L) \sum_{\sigma \in \Sigma_{\overline{M}, \max}} I(\sigma) \prod_{\substack{\mathbf{m} \in \Gamma_{\overline{M}} \\ \rho_{\mathbf{m}} \in \sigma}} \frac{1}{a_{\mathbf{m}}}.$$

The following proposition finishes the proof of Theorem 5.2.5.

Proposition 5.2.33. *For every maximal cone $\sigma' \in \Sigma$, we have*

$$\sum_{\sigma \subset \sigma'} I(\sigma) \prod_{\substack{\mathbf{m} \in \Gamma_{\overline{M}} \\ \rho_{\mathbf{m}} \in \sigma}} \frac{1}{a_{\mathbf{m}}} = \prod_{\substack{i=1 \\ \rho_i \subset \sigma'}}^{n} \frac{1}{a_i},$$

where the sum runs over all maximal cones in $\Sigma_{\overline{M}}$ contained in σ'.

Proof. Let σ be a maximal cone in $\Sigma_{\overline{M}}$. The index $I(\sigma)$ is equal to the cardinality of the quotient of $\operatorname{Pic}(X, M(\sigma))$ by the divisors $\{D_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_{M(\sigma)}, \phi(\mathbf{m}) \notin \sigma\}$. We can view this quotient as the Picard group of the pair (X, M_σ), where

$$\mathfrak{M}_\sigma = \{(0, \dots, 0)\} \cup \{\mathbf{m} \in \Gamma_{M(\sigma)} \mid \phi(\mathbf{m}) \in \sigma\}.$$

Now Proposition 5.1.3 implies that $\operatorname{Pic}(X, M_\sigma)$ is the cokernel of the homomorphism $N^\vee \to \operatorname{Div}_T(X, M_\sigma)$. As σ is a cone of dimension d, this homomorphism is an embedding of lattices of the same rank. Thus $\# \operatorname{Pic}(X, M_\sigma)$ is the index of $\operatorname{Div}_T(X, M_\sigma)^\vee$ in N, where the embedding is given by the dual of the homomorphism. As the image of a divisor $\tilde{D}_{\mathbf{m}}$ in N is $\phi(\mathbf{m})$, this implies that $\operatorname{Pic}(X, M_\sigma)$ has $|N : N_{M_\sigma}|$ elements, where we recall that N_{M_σ} is the lattice spanned by $\{\phi(\mathbf{m}) \mid \mathbf{m} \in \mathfrak{M}_\sigma\}$. We choose a basis of $N \cong \mathbb{Z}^d$ so that $\sigma' = [0, \infty)^d$ and we write $\Gamma_{M_\sigma} = \{\mathbf{m}_1, \dots, \mathbf{m}_d\}$. The set $\{\phi(\mathbf{m}) \mid \mathbf{m} \in \Gamma_{M_\sigma}\}$ is a basis of $N_{\mathbb{Q}}$, so $I(\sigma) = |N : N_{M_\sigma}|$ is equal to the absolute value of the determinant $\phi(\mathbf{m}_1) \wedge \cdots \wedge \phi(\mathbf{m}_d)$.

We will prove the desired identity by viewing both sides as an exponential integral over a cone. We order the divisors $D_1, \dots, D_n$ on X such that $\rho_1, \dots, \rho_d \in \sigma'$. Note that

$$\prod_{i=1}^{d} \frac{1}{a_i} = \int_{[0,\infty)^d} e^{-a_1 x_1 - \cdots - a_d x_d} \, \mathrm{d}x_1 \dots \mathrm{d}x_d.$$

We split up the domain of integration into the maximal cones $\sigma \in \Sigma_{\overline{M}}$ contained in $\sigma' = [0, \infty)^d$:

$$\int_{[0,\infty)^d} e^{-a_1 x_1 - \cdots - a_d x_d} \, \mathrm{d}x_1 \dots \mathrm{d}x_d = \sum_{\sigma \subset \sigma'} \int_\sigma e^{-a_1 x_1 - \cdots - a_d x_d} \, \mathrm{d}x_1 \dots \mathrm{d}x_d.$$

Now the formula [Bar93, Example 2.1] for the exponential integral over a cone gives

$$\int_\sigma e^{-a_1 x_1 - \cdots - a_d x_d} \, \mathrm{d}x_1 \dots \mathrm{d}x_d = |\phi(\mathbf{m}_1) \wedge \cdots \wedge \phi(\mathbf{m}_d)| \prod_{i=1}^{d} \frac{1}{\langle \mathbf{a}, \phi(\mathbf{m}_i) \rangle}$$

where $\mathbf{a} = (a_1, \dots, a_n)$. Since $\langle \mathbf{a}, \phi(\mathbf{m}_i) \rangle = a_{\mathbf{m}_i}$, this implies

$$\int_\sigma e^{-a_1 x_1 - \cdots - a_d x_d} \, \mathrm{d}x_1 \dots \mathrm{d}x_d = I(\sigma) \prod_{\substack{\mathbf{m} \in \Gamma_{\overline{M}} \\ \rho_{\mathbf{m}} \in \sigma}} \frac{1}{a_{\mathbf{m}}},$$

which implies the desired identity. $\square$

Samenvatting

De getaltheorie is een prominente tak van de wiskunde toegewijd aan de studie van de gehele getallen. Een belangrijk thema hierbinnen is het bestuderen van speciale verzamelingen van getallen, zoals de priemgetallen, de kwadraten, kwadraatvrije getallen en nog veel meer. Voor zo een verzameling zijn er twee belangrijke vraagstukken:

- Hoeveel zijn er, als een aandeel van alle getallen?
- Hoe zijn ze verdeeld?

Veel verzamelingen waar wiskundigen in geïnteresseerd zijn zijn oneindig, maar meestal bevatten ze maar 0% van de gehele getallen. Zo is de verzameling kwadraten $1, 4, 9, 16, 25, \ldots$ oneindig, maar is het aantal kwadraten kleiner dan een getal B ongeveer $\sqrt{B}$, veel kleiner dan B zelf. Daarentegen komen kwadraatvrije getallen $(1, 2, 3, 5, 6, 7, 9, \ldots)$ veel meer voor: het aantal kwadraatvrije getallen kleiner dan B is ongeveer $\frac{6}{\pi^2}B$. In het bijzonder is ongeveer 60,8% van de getallen kwadraatvrij.

Een van de onderwerpen in mijn proefschrift is een veralgemenisering van zulk soort telproblemen naar hogere dimensies. Dit leidt bijvoorbeeld tot vraagstukken als: hoe veel komen tripels van gehele getallen (n, m, k) voor zodat nmk een kwadraat is?

Priemontbindingen en $\mathcal{M}$-punten

Om dit soort problemen te bestuderen heb ik in mijn proefschrift de theorie van *$\mathcal{M}$-punten* geïntroduceerd. Deze theorie geeft een manier om deze problemen te bestuderen vanuit het oogpunt van de *algebraïsche meetkunde*, en is een natuurlijke uitbreiding van de theorie van Campanapunten.

Priemontbindingen spelen een belangrijke rol in dit verhaal. Elk geheel getal kan geschreven worden als een product van de priemgetallen die het getal delen. Zo hebben we $9 = 3^2$, $10 = 2 \cdot 5$ en $72 = 2^3 \cdot 3^2$. Voor een geheel getal k en een priemgetal p is de multipliciteit $v_p(k)$ van n bij p het aantal keer waarmee k door p deelbaar is. In andere woorden, het is de macht waarmee p voorkomt in de priemontbinding van k. Zo is $v_2(9) = 0$, $v_2(10) = 1$ en $v_2(72) = 3$. We definiëren de multipliciteit ook voor tupels (geordende lijsten) van gehele getallen door te stellen dat de multipliciteit van een tupel $(a_1, \ldots, a_n)$ bij p wordt gegeven door de multipliciteit van alle coördinaten samen:

$$\mathrm{mult}_p(a_1, \ldots, a_n) = (v_p(a_1), \ldots, v_p(a_n)).$$

Bijvoorbeeld, de multipliciteit van $(9, 10, 72)$ bij 2 is dus $\mathrm{mult}_2(9, 10, 72) = (0, 1, 3)$.

We kunnen dit gebruiken om $\mathcal{M}$-punten op projectieve ruimte en torische variëteiten te beschrijven. In de meetkunde zijn projectieve ruimten fundamentele objecten. Op de projectieve ruimte $\mathbb{P}^{n-1}$ zijn de (rationale) punten gegeven door de tupels $(a_1 : \cdots : a_n)$ van breuken $a_1, \ldots, a_n$, met de schalingsrelatie $(ca_1 : \cdots : ca_n) = (a_1 : \cdots : a_n)$ voor alle breuken $c \neq 0$. In het bijzonder kan elk rationaal punt geschreven worden als $(a_1 : \cdots : a_n)$, waar $a_1, \ldots, a_n$ geheel zijn en geen gemeenschappelijke deler hebben. Voor andere torische variëteiten is er een analoge beschrijving van de rationale punten, met een andere schalingsrelatie.

Neem nu een deelverzameling $\mathfrak{M} \subset \mathbb{N}^n$ van de n-tupels van natuurlijke getallen. In andere woorden, $\mathfrak{M}$ is een verzameling van tupels van de vorm $(m_1, \ldots, m_n)$, waarbij $m_1, \ldots, m_n$ allemaal natuurlijke getallen zijn. Dan is de bijbehorende collectie van $\mathcal{M}$-punten de verzameling van alle $(a_1 : \cdots : a_n)$ zodat zijn multipliciteit voor elk priemgetal p in $\mathfrak{M}$ ligt.

Bijvoorbeeld, als $n = 2$ en $\mathfrak{M}$ de verzameling is die bestaat uit $(0,0), (1,0), (0,1)$, dan zijn de $\mathcal{M}$-punten de paren (a_1, a_2) waarbij beide kwadraatvrij zijn en geen gemeenschappelijke delers hebben. Door $\mathfrak{M}$ anders te kiezen kunnnen we veel andere interessante verzamelingen krijgen, zoals de tripels (n, m, k) zodat hun product een kwadraat is en ze geen gemeenschappelijke factor samen hebben. In Sectie 2.1.4 worden nog veel andere voorbeelden gegeven.

$\mathcal{M}$-punten van begrensde hoogte

Voor een punt $(a_1, \ldots, a_n)$ op projectieve ruimte is er een natuurlijk begrip van grootte, namelijk de hoogte:

$$H(a_1 : \cdots : a_n) = \max(|a_1|, \ldots, |a_n|),$$

het maximum van de coördinaten waar mintekens buiten beschouwing worden gelaten. Dus, voor een gegeven verzameling $\mathfrak{M}$ kunnen we ons afvragen hoeveel $\mathcal{M}$-punten er zijn met hoogte kleiner dan een getal B. In mijn proefschrift heb ik bewezen in Theorem 1.2.7 dat dit aantal neigt naar $cB^a \log(B)^{b-1}$ als B naar oneindig gaat voor constanten $a, b, c > 0$. In andere woorden, als B groot is, dan is het aantal $\mathcal{M}$-punten met hoogte hooguit B ongeveer

$$cB^a \log(B)^{b-1}.$$

De constanten a en b hier zijn expliciet en hebben een meetkundige interpretatie. Als we een milde aanname doen op de verzameling $\mathfrak{M}$, kunnen we c ook expliciet uitrekenen.

Bijvoorbeeld, het aantal tripels (n, m, k) van gehele getallen zodat nmk kwadraatvol is en ze alledrie kleiner dan B zijn is ongeveer gelijk aan $1{,}724\sqrt{B^3}\log(B)^3$. (Een getal is kwadraatvol als het geschreven kan worden als een product van een kwadraat en een derde macht, of equivalent: de multipliciteit van het getal bij elk priemgetal is ongelijk aan 1.)

Algemener kunnen op andere variëteiten (ruimtes beschreven door vergelijkingen) ook hoogtes en $\mathcal{M}$-punten worden geformuleerd. Met deze hoogtes geeft Theorem 1.2.7 ook een beschrijving van het aantal $\mathcal{M}$-punten van begrensde hoogte op (gespleten) torische variëteiten. Deze resultaten breiden de resultaten van Pieropan en Schindler [PS24a] voor torische variëteiten uit van Campanapunten tot de veel algemenere setting van $\mathcal{M}$-punten. In het bijzonder geeft Theorem 1.2.7 ook een beschrijving van het aantal zogehete zwakke Campanapunten van begrensde hoogte, zoals bijvoorbeeld het gegeven voorbeeld hierboven. Dit was al door eerdere auteurs bestudeerd, maar die hadden geen resultaten of voorspellingen voor hoeveel er zijn.

Verspreiding van $\mathcal{M}$-punten

Een ander onderwerp dat ik heb bestudeerd in mijn proefschrift is de verdeling van $\mathcal{M}$-punten op projectieve ruimte en andere torische variëteiten. Hiervoor werken we met modulorekenen. Voor gehele getallen n, m is $n \mod m$ het unieke gehele getal k tussen 0 en $m-1$ waarvoor we $n = am + k$ kunnen schrijven, waar a geheel is. Oftewel $n \bmod m$ is de rest na deling door m. We zijn geïnteresseerd in de volgende vraag: gegeven een macht van een priemgetal p, zijn alle tupels $(b_1, \ldots, b_n)$ met $0 < b_1, \ldots, b_n < m$ van de vorm $(a_1 \bmod m, \ldots, a_n \bmod m)$ voor een $\mathcal{M}$-punt $(a_1 : \cdots : a_n)$? Als dit kan, dan zeggen we dat $\mathcal{M}$-benadering geldt.

In het algemeen is dit niet het geval. Zo kan dit niet gedaan worden op de projectieve lijn $\mathbb{P}^1$ met $\mathfrak{M}$ gegeven door de paren van even getallen. Voor deze keuze van $\mathfrak{M}$ zijn de $\mathcal{M}$-punten gegeven door tupels van de vorm $(a^2 : b^2)$ of $(-a^2 : b^2)$ met a en b gehele getallen, en het paar $(2, 1)$ is niet te schrijven als $(a^2 \bmod 5, b^2 \bmod 5)$ of $(-a^2 \bmod 5, b^2 \bmod 5)$. In mijn proefschrift heb ik een manier gevonden om eenvoudig te bepalen of $\mathcal{M}$-benadering geldt voor projectieve ruimte of in algemener een (gespleten) torische variëteit, gegeven in Theorem 1.1.3. Dit is een brede uitbreiding van het recente werk van Nakahara en Streeter [NS24], die aantoonden dat $\mathcal{M}$-benadering geldt in de beperktere setting van Campanapunten.

Acknowledgements

There are many lovely people I would like to thank for their support and friendship during my PhD.

First of all I would like to thank my supervisor, Marta Pieropan. I thank you for your guidance and support through these years. You allowed me to find my own topics to explore, while you were always available when I needed your help. I find it inspiring how you are very active in organizing and participating in many events, such as the number theory and geometry seminar, Women in Numbers, EWM and more. I am very grateful for your supervision though these years, which I will fondly look back on. I would also like to thank my promotor, Carel Faber. While we have not spoken to each other that often, your guidance though these years has been highly appreciated.

I would like to thank the reading committee for their time to read my thesis, and I would especially like to thank Gunter for his many useful suggestions and corrections.

There are many great people I met over the years travelling to conferences, of which I have plenty of nice memories. I especially would like to thank Sam and Alec for the nice discussions we've had about mathematics and for the workshop in Bristol they organized. I still recall the nice talk we've had when walking in nature around Luminy. I would also like to thank the wonderful people in Bath, including Harry, Elyes and Dan for their invitation to speak at their seminar and for graciously funding my stay during my week there. Vlad, I would also like to thank you for inviting me over to Vratsa and Sofia last September, and for the terrific conference you organized in March.

I would also like to thank my fellow PhD students and other colleagues in Utrecht. In particular, I thank my office mates Dušan, Maite, Marco and Reinier, for making the office a nice place to be during these years. There have been many cases where you have helped me during my journey. I appreciate the discussions we have had about mathematics, but I have especially appreciated the many conversations about different topics. Reinier, I also really appreciate the board game evenings you organized over the years, which were always fun to participate in.

Furthermore, I thank the people in the number theory and algebraic geometry groups for the many nice seminars and the social activities, such as going for dinner or going to the Keukenhof together. I would especially like to thank Sara for her amazing work in designing and drawing the cover of this thesis.

I would be remiss not to thank the wonderful people in the rational points consortium in Groningen and Leiden. The meetings and activities we've had were always nice to be at. I fondly recall the barbecues we had together at Martin's house, the boat trip through the light festival in Amsterdam, and the dinner we had at Haowen's house.

I would also like to thank Justin, Reinier, Haowen, Sebastián and Maud for their friendship during these years. Justin, it has been fun having you as an academic brother, and it was always nice to see you together with Sophie when she came over to visit. Reinier, thank you for the many nice conversations we had during our bike rides home after a day at the office. Haowen, I really love your radiant personality, which always improves my mood. Sebastián, I like how you are always up to play games with me, both in person as well as online. Thank you Maud for the many nice concerts we have shared together.

Last, but certainly not least, I would like to thank my parents and my brother for their unwavering support during these years, as well as before. You are always there for me, during nice periods as well as tough ones. With your help, I have been able to get where I am now, for which I am very grateful.

Curriculum Vitae

Boaz Constantijn Moerman was born on April 21st 1998 in Nieuwegein, near Utrecht, and grew up in Nijmegen, where he went to primary and secondary school. In 2016, Boaz graduated from Nijmeegse Scholengemeenschap Groenewoud. Inspired by his mathematics teacher Martin Winkel, he decided to study mathematics at the Radboud University, where he obtained his bachelor's degree in 2019. In this period, he also participated in the honours program at the Radboud. As part of this, he wrote his extended bachelor thesis "Eisenstein series and periods" under the supervision of Professor Wadim Zudilin. He continued his studies at the Radboud university, and obtained a master's degree in Mathematics in 2021. During his master's studies, he wrote his master thesis "Genus 2 fibrations" under the supervision of Professor Ben Moonen. From September 2021 until August 2025, he worked as a PhD candidate at Utrecht University under the supervision of Dr. Marta Pieropan, which resulted in this thesis. Boaz's future is yet to be decided.

Bibliography

[Abr09] Dan Abramovich. Birational geometry for number theorists. In *Arithmetic geometry*. Volume 8, Clay Math. Proc. Pages 335–373. Amer. Math. Soc., Providence, RI, 2009. ISBN: 978-0-8218-4476-2.

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013. ISSN: 0373-0956,1777-5310. DOI: `10.5802/aif.2833`.

[AP13] Valerii A. Alexeev and Rita Pardini. On the existence of ramified abelian covers. *Rend. Semin. Mat. Univ. Politec. Torino*, 71(3-4):307–315, 2013. ISSN: 0373-1243,2704-999X.

[Art67] Emil Artin. *Algebraic numbers and algebraic functions*. Gordon and Breach Science Publishers, New York-London-Paris, 1967, pages xiii+349. ISBN: 9780677006352.

[AV18] Dan Abramovich and Anthony Várilly-Alvarado. Campana points, Vojta's conjecture, and level structures on semistable abelian varieties. *J. Théor. Nombres Bordeaux*, 30(2):525–532, 2018. ISSN: 1246-7405,2118-8572.

[Bar93] Alexander I. Barvinok. Computing the volume, counting integral points, and exponential sums. *Discrete Comput. Geom.*, 10(2):123–141, 1993. ISSN: 0179-5376,1432-0444. DOI: `10.1007/BF02573970`.

[BBK+24] Francesca Balestrieri, Julia Brandes, Miriam Kaesberg, Judith Ortmann, Marta Pieropan, and Rosa Winter. Campana points on diagonal hypersurfaces. In *Women in numbers Europe IV—research directions in number theory*. Volume 32, Assoc. Women Math. Ser. Pages 63–92. Springer, Cham, 2024. DOI: `10.1007/978-3-031-52163-8\_3`.

[BC08] Constantin N. Beli and Wai Kiu Chan. Strong approximation of quadrics and representations of quadratic forms. *J. Number Theory*, 128(7):2091–2096, 2008. ISSN: 0022-314X,1096-1658. DOI: `10.1016/j.jnt.2007.07.010`.

[BCD+01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over **Q**: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001. ISSN: 0894-0347,1088-6834. DOI: `10.1090/S0894-0347-01-00370-8`.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-09-00649-3.

[BCJ+24] Finn Bartsch, Frédéric Campana, Ariyan Javanpeykar, and Olivier Wittenberg. The Weakly Special Conjecture contradicts orbifold Mordell, and thus abc, 2024. arXiv: 2410.06643.

[BD13] Mikhail Borovoi and Cyril Demarche. Manin obstruction to strong approximation for homogeneous spaces. *Comment. Math. Helv.*, 88(1):1–54, 2013. ISSN: 0010-2571,1420-8946. DOI: 10.4171/CMH/277.

[BFP14] Lior Bary-Soroker, Arno Fehm, and Sebastian Petersen. On varieties of Hilbert type. *Ann. Inst. Fourier (Grenoble)*, 64(5):1893–1901, 2014. ISSN: 0373-0956,1777-5310.

[BFP23] Lior Bary-Soroker, Arno Fehm, and Sebastian Petersen. Hilbert properties under base change in small extensions, 2023. arXiv: 2312.16219.

[BJ24] Finn Bartsch and Ariyan Javanpeykar. Kobayashi-Ochiai's finiteness theorem for orbifold pairs of general type. *J. Inst. Math. Jussieu*, 23(6):2713–2732, 2024. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474748024000094.

[BJR24] Finn Bartsch, Ariyan Javanpeykar, and Erwan Rousseau. Weakly-special threefolds and non-density of rational points, 2024. arXiv: 2310.09065.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990. ISBN: 3-540-50587-3. DOI: 10.1007/978-3-642-51438-8.

[BM90] Victor V. Batyrev and Yuri I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01453564.

[BMO+18] Michael A. Bennett, Greg Martin, Kevin O'Bryant, and Andrew Rechnitzer. Explicit bounds for primes in arithmetic progressions. *Illinois J. Math.*, 62(1-4):427–532, 2018. ISSN: 0019-2082,1945-6581. DOI: 10.1215/ijm/1552442669.

[BS13] Michel Brion and Tamás Szamuely. Prime-to-p étale covers of algebraic groups and homogeneous spaces. *Bull. Lond. Math. Soc.*, 45(3):602–612, 2013. ISSN: 0024-6093,1469-2120. DOI: 10.1112/blms/bds110.

[BT95] Victor V. Batyrev and Yuri Tschinkel. Rational points of bounded height on compactifications of anisotropic tori. *Internat. Math. Res. Notices*, (12):591–635, 1995. ISSN: 1073-7928,1687-0247. DOI: 10.1155/S1073792895000365.

[BT96] Victor V. Batyrev and Yuri Tschinkel. Height zeta functions of toric varieties. In volume 82, number 1, pages 3220–3239. 1996. DOI: 10.1007/BF02362469.

[BY21] Tim Browning and Shuntaro Yamagishi. Arithmetic of higher-dimensional orbifolds and a mixed Waring problem. *Math. Z.*, 299(1-2):1071–1101, 2021. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-021-02695-w.

[Cad07] Charles Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007. ISSN: 0002-9327,1080-6377. DOI: 10.1353/ajm.2007.0007.

[Cam05] Frédéric Campana. Fibres multiples sur les surfaces: aspects geométriques, hyperboliques et arithmétiques. *Manuscripta Math.*, 117(4):429–461, 2005. ISSN: 0025-2611,1432-1785. DOI: 10.1007/s00229-005-0570-5.

[Cam11a] Frédéric Campana. Orbifoldes géométriques spéciales et classification biméromorphe des variétés kählériennes compactes. *J. Inst. Math. Jussieu*, 10(4):809–934, 2011. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474748010000101.

[Cam11b] Frédéric Campana. Special orbifolds and birational classification: a survey. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep. Pages 123–170. Eur. Math. Soc., Zürich, 2011. ISBN: 978-3-03719-007-4. DOI: 10.4171/007-1/6.

[Cao18] Yang Cao. Approximation forte pour les variétés avec une action d'un groupe linéaire. *Compos. Math.*, 154(4):773–819, 2018. ISSN: 0010-437X,1570-5846. DOI: 10.1112/S0010437X17007709.

[CF67] John W. S. Cassels and Albrecht Fröhlich, editors. *Algebraic number theory*, 1967. Academic Press, London; Thompson Book Co., Inc., Washington, DC, pages xviii+366. ISBN: 978-0-9502734-2-6.

[CH16] Jean-Louis Colliot-Thélène and David Harari. Approximation forte en famille. *J. Reine Angew. Math.*, 710:173–198, 2016. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2013-0092.

[CH20] Yang Cao and Zhizhong Huang. Arithmetic purity of strong approximation for semi-simple simply connected groups. *Compos. Math.*, 156(12):2628–2649, 2020. ISSN: 0010-437X,1570-5846. DOI: 10.1112/s0010437x20007617.

[Che24] Sheng Chen. Arithmetic purity of strong approximation for complete toric varieties. *Math. Z.*, 306(2):23, 2024. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-023-03418-z.

[Chr20] Atticus Christensen. A topology on points on stacks, 2020. arXiv: 2005.10231.

[CL14] Yang Cao and Yongqi Liang. Local-global principle for certain biquadratic normic bundles. *Acta Arith.*, 164(2):137–144, 2014. ISSN: 0065-1036,1730-6264. DOI: 10.4064/aa164-2-3.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011, pages xxiv+841. ISBN: 978-0-8218-4819-7. DOI: 10.1090/gsm/124.

[CLT+25] Dylon Chow, Daniel Loughran, Ramin Takloo-Bighash, and Sho Tanimoto. Campana points on wonderful compactifications, 2025. arXiv: 2403.14433.

[CLT24] Qile Chen, Brian Lehmann, and Sho Tanimoto. Campana rational connectedness and weak approximation, 2024. arXiv: 2406.04991.

[CLX19] Yang Cao, Yongqi Liang, and Fei Xu. Arithmetic purity of strong approximation for homogeneous spaces. *J. Math. Pures Appl. (9)*, 132:334–368, 2019. ISSN: 0021-7824,1776-3371. DOI: 10.1016/j.matpur.2019.02.018.

[Col18] Jean-Louis Colliot-Thélène. Approximation forte pour les espaces homogènes de groupes semi-simples sur le corps des fonctions d'une courbe algébrique complexe. *Eur. J. Math.*, 4(1):177–184, 2018. ISSN: 2199-675X,2199-6768. DOI: 10.1007/s40879-017-0133-9.

[Col88] Jean-Louis Colliot-Thélène. Letter to T.Ekedahl, 1988. unpublished.

[Con06] Brian Conrad. Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *Enseign. Math. (2)*, 52(1-2):37–108, 2006. ISSN: 0013-8584.

[Con12] Brian Conrad. Weil and Grothendieck approaches to adelic points. *Enseign. Math. (2)*, 58(1-2):61–97, 2012. ISSN: 0013-8584. DOI: 10.4171/LEM/58-1-3.

[Cox22] David A. Cox. *Primes of the form $x^2 + ny^2$—Fermat, class field theory, and complex multiplication.* AMS Chelsea Publishing, Providence, RI, third edition, 2022. ISBN: 9781470470289.

[Cox95] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995. ISSN: 1056-3911,1534-7486.

[CS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer, Cham, 2021, pages xv+453. ISBN: 978-3-030-74248-5. DOI: 10.1007/978-3-030-74248-5.

[CS89] Jean-Louis Colliot-Thélène and Per Salberger. Arithmetic on some singular cubic hypersurfaces. *Proc. London Math. Soc. (3)*, 58(3):519–549, 1989. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms/s3-58.3.519.

[CSS87] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Peter Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. I. *J. Reine Angew. Math.*, 373:37–107, 1987. ISSN: 0075-4102,1435-5345.

[CT10] Antoine Chambert-Loir and Yuri Tschinkel. Igusa integrals and volume asymptotics in analytic and adelic geometry. *Confluentes Math.*, 2(3):351–429, 2010. ISSN: 1793-7442,1793-7434. DOI: 10.1142/S1793744210000223.

[CX18a] Yang Cao and Fei Xu. Strong approximation with Brauer-Manin obstruction for groupic varieties. *Proc. Lond. Math. Soc. (3)*, 117(4):727–750, 2018. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms.12141.

[CX18b] Yang Cao and Fei Xu. Strong approximation with Brauer-Manin obstruction for toric varieties. *Ann. Inst. Fourier (Grenoble)*, 68(5):1879–1908, 2018. ISSN: 0373-0956,1777-5310.

[CZ18] Qile Chen and Yi Zhu. Strong approximation over function fields. *J. Algebraic Geom.*, 27(4):703–725, 2018. ISSN: 1056-3911,1534-7486. DOI: 10.1090/jag/706.

[Dar97] Henri Darmon. Faltings plus epsilon, Wiles plus epsilon, and the generalized Fermat equation. *C. R. Math. Rep. Acad. Sci. Canada*, 19(1):3–14, 1997. ISSN: 0706-1994.

[Dem22] Julian L. Demeio. The étale Brauer-Manin obstruction to strong approximation on homogeneous spaces. *Trans. Amer. Math. Soc.*, 375(12):8581–8634, 2022. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8705.

[DF93] Frank R. DeMeyer and Timothy J. Ford. On the Brauer group of toric varieties. *Trans. Amer. Math. Soc.*, 335(2):559–577, 1993. ISSN: 0002-9947,1088-6850. DOI: 10.2307/2154394.

[dlBre01a] Régis de la Bretèche. Compter des points d'une variété torique. *J. Number Theory*, 87(2):315–331, 2001. ISSN: 0022-314X,1096-1658. DOI: 10.1006/jnth.2000.2605.

[dlBre01b] Régis de la Bretèche. Estimation de sommes multiples de fonctions arithmétiques. *Compositio Math.*, 128(3):261–298, 2001. ISSN: 0010-437X,1570-5846. DOI: 10.1023/A:1011803816545.

[DW17] Ulrich Derenthal and Dasheng Wei. Strong approximation and descent. *J. Reine Angew. Math.*, 731:235–258, 2017. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2014-0149.

[DY24] Ratko Darda and Takehiko Yasuda. The Batyrev-Manin conjecture for DM stacks, 2024. arXiv: 2207.03645.

[EGA4] Alexander Grothendieck. Éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28), 1966. ISSN: 0073-8301,1618-1913.

[Eic38] Martin Eichler. Allgemeine Kongruenzklasseneinteilungen der Ideale einfacher Algebren über algebraischen Zahlkörper und ihre L-Reihen. *J. Reine Angew. Math.*, 179:227–251, 1938. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crll.1938.179.227.

[ES34] Pál Erdős and George Szekeres. On the number of abelian groups of given order and on a related number-theoretic problem. German. *Acta Litt. Sci. Szeged*, 7:95–102, 1934.

[Ess07] Driss Essouabri. Manin's conjecture on toric varieties with different heights, 2007. arXiv: 0711.1520.

[ESZ23] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *Forum Math. Sigma*, 11:Paper No. e14, 54, 2023. ISSN: 2050-5094. DOI: 10.1017/fms.2023.5.

[EW87] Richard Elman and Adrian R. Wadsworth. Hereditarily quadratically closed fields. *J. Algebra*, 111(2):475–482, 1987. ISSN: 0021-8693. DOI: 10.1016/0021-8693(87)90231-6.

[Fai23] Loïs Faisant. *Stabilisation phenomena in moduli spaces of curves*. Thesis, Université Grenoble Alpes, 2023. URL: https://theses.hal.science/tel-04250474.

[Fai25] Loïs Faisant. Motivic counting of rational curves with tangency conditions via universal torsors, 2025. arXiv: 2502.11704.

[Fal83] Gerd Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01388432.

[Fil92] Michael Filaseta. Squarefree values of polynomials. *Acta Arith.*, 60(3):213–231, 1992. ISSN: 0065-1036. DOI: 10.4064/aa-60-3-213-231.

[FJ05] Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2005. ISBN: 3-540-22811-X.

[FMT89] Jens Franke, Yuri I. Manin, and Yuri Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01393904.

[FP16] Christopher Frei and Marta Pieropan. O-minimality on twisted universal torsors and Manin's conjecture over number fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):757–811, 2016. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2295.

[Har20] David Harari. *Galois cohomology and class field theory*. Universitext. Springer, Cham, 2020, pages xiv+338. ISBN: 978-3-030-43901-9. DOI: 10.1007/978-3-030-43901-9.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977. ISBN: 0-387-90244-9.

[Har95] David Harari. Principe de Hasse et approximation faible sur certaines hypersurfaces. *Ann. Fac. Sci. Toulouse Math. (6)*, 4(4):731–762, 1995. ISSN: 0240-2955. DOI: 10.5802/afst.810.

[Hea90] Roger Heath-Brown. Sums of three square-full numbers. In *Number theory, Vol. I (Budapest, 1987)*. Volume 51, Colloq. Math. Soc. János Bolyai, pages 163–171. North-Holland, Amsterdam, 1990. ISBN: 0-444-70346-2.

[Hoo67] Christopher Hooley. On the power free values of polynomials. *Mathematika*, 14:21–26, 1967. ISSN: 0025-5793. DOI: 10.1112/S002557930000797X.

[HS00] Marc Hindry and Joseph H. Silverman. *Diophantine geometry*, volume 201 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000, pages xiv+558. ISBN: 0-387-98975-7. DOI: 10.1007/978-1-4612-1210-2.

[HT00] Joe Harris and Yuri Tschinkel. Rational points on quartics. *Duke Math. J.*, 104(3):477–500, 2000. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-00-10436-X.

[HT06] Brendan Hassett and Yuri Tschinkel. Weak approximation over function fields. *Invent. Math.*, 163(1):171–190, 2006. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-005-0458-8.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992. ISSN: 1056-3911,1534-7486.

[Kne66] Martin Kneser. Strong approximation. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 187–196. Amer. Math. Soc., Providence, RI, 1966.

[KPS22] Stefan Kebekus, Jorge V. Pereira, and Arne Smeets. Failure of the Brauer-Manin principle for a simply connected fourfold over a global function field, via orbifold Mordell. *Duke Math. J.*, 171(17):3515–3591, 2022. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2022-0045.

[Kum14] Manish Kumar. The fundamental group of affine curves in positive characteristic. *J. Algebra*, 399:323–342, 2014. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2013.10.007.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002, pages xvi+914. ISBN: 0-387-95385-X. DOI: 10.1007/978-1-4613-0041-0.

[Lan03] Edmund Landau. Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes. *Math. Ann.*, 56(4):645–670, 1903. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01444310.

[Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994, pages xiv+357. ISBN: 0-387-94225-4. DOI: 10.1007/978-1-4612-0853-2.

[Liu06] Qing Liu. *Algebraic geometry and arithmetic curves.* Volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2006. ISBN: 0-19-920249-4.

[LS16] Oliver Lorscheid and Cecília Salgado. A remark on topologies for rational point sets. *J. Number Theory*, 159:193–201, 2016. ISSN: 0022-314X,1096-1658. DOI: 10.1016/j.jnt.2015.07.015.

[LSS20] Daniel Loughran, Alexei N. Skorobogatov, and Arne Smeets. Pseudo-split fibers and arithmetic surjectivity. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(4):1037–1070, 2020. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2439.

[LST22] Brian Lehmann, Akash Kumar Sengupta, and Sho Tanimoto. Geometric consistency of Manin's conjecture. *Compos. Math.*, 158(6):1375–1427, 2022. ISSN: 0010-437X,1570-5846. DOI: 10.1112/s0010437x22007588.

[Luc14] G. Lucchini Arteche. Approximation faible et principe de Hasse pour des espaces homogènes à stabilisateur fini résoluble. *Math. Ann.*, 360(3-4):1021–1039, 2014. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-014-1058-y.

[Lug22] Cedric Luger. The Hilbert property for arithmetic schemes. *Acta Arith.*, 206(2):115–126, 2022. ISSN: 0065-1036,1730-6264. DOI: 10.4064/aa211214-16-11.

[LW54] Serge Lang and André Weil. Number of points of varieties in finite fields. *Amer. J. Math.*, 76:819–827, 1954. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2372655.

[Mar77] Grigory A. Margulis. Cobounded subgroups in algebraic groups over local fields. *Funkcional. Anal. i Priložen.*, 11(2):45–57, 95, 1977. ISSN: 0374-1990.

[Min89] Hristo P. Minchev. Strong approximation for varieties over an algebraic number field. *Dokl. Akad. Nauk BSSR*, 33(1):5–8, 92, 1989. ISSN: 0002-354X.

[MNS24] Vladimir Mitankin, Masahiro Nakahara, and Sam Streeter. Semi-integral Brauer-Manin obstruction and quadric orbifold pairs. *Trans. Amer. Math. Soc.*, 377(6):4435–4480, 2024. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/9170.

[Mor20] Laurent Moret-Bailly. Points rationnels dans leur fibre: compléments à un théorème de Poonen. *J. Théor. Nombres Bordeaux*, 32(2):471–488, 2020. ISSN: 1246-7405,2118-8572. DOI: 10.5802/jtnb.1130.

[Nar04] Władysław Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, third edition, 2004, pages xii+708. ISBN: 3-540-21902-1. DOI: 10.1007/978-3-662-07001-7.

[Noo04] Behrang Noohi. Fundamental groups of algebraic stacks. *J. Inst. Math. Jussieu*, 3(1):69–103, 2004. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474748004000039.

[NR24] Masahiro Nakahara and Samuel Roven. Weak approximation on Châtelet surfaces. *Manuscripta Math.*, 174(3-4):1133–1149, 2024. ISSN: 0025-2611,1432-1785. DOI: 10.1007/s00229-024-01548-0.

[NS24] Masahiro Nakahara and Sam Streeter. Weak approximation and the Hilbert property for Campana points. *Michigan Math. J.*, 74(2):227–252, 2024. ISSN: 0026-2285,1945-2365. DOI: 10.1307/mmj/20216051.

[Ogu18] Arthur Ogus. *Lectures on logarithmic algebraic geometry*, volume 178 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018, pages xviii+539. ISBN: 978-1-107-18773-3. DOI: 10.1017/9781316941614.

[Pey95] Emmanuel Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-95-07904-6.

[Pla69] Vladimir P. Platonov. The problem of strong approximation and the Kneser-Tits hypothesis for algebraic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:1211–1219, 1969. ISSN: 0373-2436.

[Poo03] Bjorn Poonen. Squarefree values of multivariable polynomials. *Duke Math. J.*, 118(2):353–373, 2003. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-03-11826-8.

[Poo17] Bjorn Poonen. *Rational points on varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017, pages xv+337. ISBN: 978-1-4704-3773-2. DOI: 10.1090/gsm/186.

[Pop96] Florian Pop. Embedding problems over large fields. *Ann. of Math. (2)*, 144(1):1–34, 1996. ISSN: 0003-486X,1939-8980. DOI: 10.2307/2118581.

[Pra77] Gopal Prasad. Strong approximation for semi-simple groups over function fields. *Ann. of Math. (2)*, 105(3):553–572, 1977. ISSN: 0003-486X. DOI: 10.2307/1970924.

[PS24a] Marta Pieropan and Damaris Schindler. Hyperbola method on toric varieties. *J. Éc. polytech. Math.*, 11:107–157, 2024. ISSN: 2429-7100,2270-518X. DOI: 10.5802/jep.251.

[PS24b] Marta Pieropan and Damaris Schindler. Points of bounded height on certain subvarieties of toric varieties, 2024. arXiv: 2403.19397.

[PSTVA21] Marta Pieropan, Arne Smeets, Sho Tanimoto, and Anthony Várilly-Alvarado. Campana points of bounded height on vector group compactifications. *Proc. Lond. Math. Soc. (3)*, 123(1):57–101, 2021. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms.12391.

[QR22] Ming Hao Quek and David Rydh. Weighted blow-ups, 2022. URL: https://people.kth.se/~dary/weighted-blowups20220329.pdf. In preparation.

[Ryd13] David Rydh. Existence and properties of geometric quotients. *J. Algebraic Geom.*, 22(4):629–669, 2013. ISSN: 1056-3911,1534-7486. DOI: 10.1090/S1056-3911-2013-00615-3.

[Sal98] Per Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque*, (251):91–258, 1998. ISSN: 0303-1179,2492-5926.

[San23a] Tim Santens. Manin's conjecture for integral points on toric varieties, 2023. arXiv: 2312.13914.

[San23b] Tim Santens. The Brauer-Manin obstruction for stacky curves, 2023. arXiv: 2210.17184.

[Ser08] Jean-Pierre Serre. *Topics in Galois theory*, volume 1 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, second edition, 2008, pages xvi+120. ISBN: 978-1-56881-412-4.

[SGA1] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[Shu21] Alec Shute. Sums of four squareful numbers, 2021. arXiv: 2104.06966.

[Shu22] Alec Shute. On the leading constant in the Manin-type conjecture for Campana points. *Acta Arith.*, 204(4):317–346, 2022. ISSN: 0065-1036,1730-6264. DOI: 10.4064/aa210430-1-7.

[Ski97] Christopher M. Skinner. Forms over number fields and weak approximation. *Compositio Math.*, 106(1):11–29, 1997. ISSN: 0010-437X,1570-5846. DOI: 10.1023/A:1000129818730.

[Sko01] Alexei N. Skorobogatov. *Torsors and rational points*, volume 144 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001, pages viii+187. ISBN: 0-521-80237-7. DOI: 10.1017/CBO9780511549588.

[Sme17] Arne Smeets. Insufficiency of the étale Brauer-Manin obstruction: towards a simply connected example. *Amer. J. Math.*, 139(2):417–431, 2017. ISSN: 0002-9327,1080-6377. DOI: 10.1353/ajm.2017.0010.

[Spr98] Maria L. Spreafico. Axiomatic theory for transversality and Bertini type theorems. *Arch. Math. (Basel)*, 70(5):407–424, 1998. ISSN: 0003-889X,1420-8938. DOI: 10.1007/s000130050213.

[SS24] Alec Shute and Sam Streeter. Semi-integral points of bounded height on toric varieties, 2024. arXiv: 2410.02039.

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/.

[Str22] Sam Streeter. Campana points and powerful values of norm forms. *Math. Z.*, 301(1):627–664, 2022. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-021-02922-4.

[SW23] Gian C. Sanjaya and Xiaoheng Wang. On the squarefree values of a^4+b^3. *Math. Ann.*, 386(3-4):1237–1265, 2023. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-022-02404-w.

[Ten15] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, french edition, 2015, pages xxiv+629. ISBN: 978-0-8218-9854-3. DOI: 10.1090/gsm/163.

[Van12] Karl Van Valckenborgh. Squareful numbers in hyperplanes. *Algebra Number Theory*, 6(5):1019–1041, 2012. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2012.6.1019.

[Vár08] Anthony Várilly-Alvarado. Weak approximation on del Pezzo surfaces of degree 1. *Adv. Math.*, 219(6):2123–2145, 2008. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2008.08.007.

[Voj87] Paul Vojta. *Diophantine approximations and value distribution theory*, volume 1239 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. ISBN: 3-540-17551-2. DOI: 10.1007/BFb0072989.

[VZ22] John Voight and David Zureick-Brown. The canonical ring of a stacky curve. *Mem. Amer. Math. Soc.*, 277(1362):v+144, 2022. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/1362.

[Wan96] Lan Wang. Brauer-Manin obstruction to weak approximation on abelian varieties. *Israel J. Math.*, 94:189–200, 1996. ISSN: 0021-2172,1565-8511. DOI: 10.1007/BF02762704.

[Wei21] Dasheng Wei. Strong approximation for a toric variety. *Acta Math. Sin. (Engl. Ser.)*, 37(1):95–103, 2021. ISSN: 1439-8516,1439-7617. DOI: 10.1007/s10114-021-8193-7.

[Wei28] André Weil. *L'arithmétique sur les courbes algébriques*. Thesis, Université de Paris, 1928.

[WX12] Dasheng Wei and Fei Xu. Integral points for multi-norm tori. *Proc. Lond. Math. Soc. (3)*, 104(5):1019–1044, 2012. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms/pdr068.

[Xia22] Huan Xiao. Campana points on biequivariant compactifications of the Heisenberg group. *Eur. J. Math.*, 8(1):205–246, 2022. ISSN: 2199-675X,2199-6768. DOI: 10.1007/s40879-021-00498-1.

[Xu15] Fei Xu. Strong approximation for certain quadric fibrations with compact fibers. *Adv. Math.*, 281:279–295, 2015. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2015.02.020.

[Yam02] Aiichi Yamasaki. Toward a generalization of strong approximation theorem to a general PF field. *J. Math. Kyoto Univ.*, 42(3):477–484, 2002. ISSN: 0023-608X. DOI: 10.1215/kjm/1250283845.

[Yam96] Aiichi Yamasaki. Cancellation of lattices and approximation properties of division algebras. *J. Math. Kyoto Univ.*, 36(4):857–867, 1996. ISSN: 0023-608X. DOI: 10.1215/kjm/1250518456.

[ZW12] Wenpeng Zhang and Tingting Wang. Powerful numbers in $(1^k+1)(2^k+1)\cdots(n^k+1)$. *J. Number Theory*, 132(11):2630–2635, 2012. ISSN: 0022-314X,1096-1658. DOI: 10.1016/j.jnt.2012.05.025.