Counting M-points of bounded height on toric varieties

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Overview

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In this talk, I will give a brief introduction to Manin's conjecture on the number of rational points of bounded height.
After this, I will introduce an extension of Manin's conjecture to collections of special rational points (*M-points*), which I have proven for toric varieties. This will provide a framework to understand the distribution of special solutions of polynomial equations (squarefree, squareful, coprime etc).

Heights

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$$H(P) := \max(|a_1|, \ldots, |a_n|).$$

Heights

More generally, if X is a projective variety with a morphism $i\colon X \to \mathbb{P}^{n-1}$ to projective space, we obtain a height

$$H_L\colon X(\mathbb{Q}) \to \mathbb{R}_{>0}$$

by $H_L(P) := H(i(P))$. Here $L \in \mathrm{Pic}(X)$ denotes the pullback of the class of the hyperplane divisor along i: $L = i^*[\{X_1 = 0\}]$.

Counting points on varieties

Using the height H_L on X, we want to understand the asymptotics of the counting function

$$N_{X,L}(B) := \#\{P \in X(\mathbb{Q}) \setminus Z \mid H_L(P) \leq B\}.$$

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Here $Z \subset X(\mathbb{Q})$ is some *thin set*: a set which lies in the union of a closed set C and images $f_1(Y_1(\mathbb{Q})), \ldots, f_n(Y_n(\mathbb{Q}))$, where $f_i \colon Y_i \to X$ are finite morphisms.

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Many varieties are rationally connected, including toric varieties as well as Fano varieties such as hypersurfaces

$$\{f = 0\} \subset \mathbb{P}^d,$$

where f has degree at most d.

Manin's conjecture

We state a modern version of Manin's conjecture. Let $\mathrm{Eff}(X) \subset \mathrm{Pic}(X) \otimes \mathbb{R}$ be effective cone: the set of nonnegative linear combinations of prime divisors and let K_X be the canonical divisor class.

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Let X be a smooth projective rationally connected variety and let L be a big and nef (e.g. ample) line bundle on X. Then we have

$$N_{X,L}(B) \sim cB^a(\log B)^{b-1} \quad \text{as } B \to \infty,$$

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$$N_{X,L}(B) \sim cB^a(\log B)^{b-1} \quad \text{as } B \to \infty,$$

where

$$a = \inf\{t \in \mathbb{R} \mid tL + K_X \in \mathrm{Eff}(X)\},$$

and b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $aL + K_X$.

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We now illustrate this result with an example.

Example: product of projective lines

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, with coordinates $((X : Y), (Z : W))$. Then $\mathrm{Pic}(X) = \mathbb{Z}^2$, generated by $D_1 = \{X = 0\}$ and $D_2 = \{Z = 0\}$, and $K_X = 2[D_1 + D_2]$.

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Under the Segre embedding $i \colon X \to \mathbb{P}^3$,

$$((a : b), (c : d)) \mapsto (ac : ad : bc : bd),$$

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On the other hand, $i \colon X \to \mathbb{P}^5$ given by

$$((a : b), (c : d)) \mapsto (a^2c : a^2d : abc : abd : b^2c : b^2d)$$

gives $L = -\frac{1}{2}K_X + [D_1]$ so

$$N_{X,L}(B) \sim c'B^2.$$

Rather than counting the full set of rational points, we can also count in interesting subsets. For instance, we would like to count the points on projective space for which the product of the coordinates is squareful. (An integer is squareful if it can be written as the product of a square and a cube.)

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For instance, my results will imply that the number of triples $(x : y : z) \in \mathbb{P}^2(\mathbb{Z})$ such that xyz is squareful and $\max(|x|, |y|, |z|) \leq B$ is asymptotic to

$$cB^{3/2}(\log B)^3$$

as $B \to \infty$.

Multiplicities

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$$\mathrm{mult}_p \colon X(\mathbb{Q}) \to (\mathbb{N} \cup \{\infty\})^n$$

$$Q \mapsto (n_p(D_1, Q), \ldots, n_p(D_n, Q)).$$

Multiplicities on projective space

For example, if $X = \mathbb{P}^{n-1}$ and $D_1, \ldots, D_n$ are the coordinate hyperplanes, then the multiplicity is simply the tuple of valuations

$$\mathrm{mult}_p(a_1 : \cdots : a_n) = (v_p(a_1), \ldots, v_p(a_n))$$

if $\gcd(a_1, \ldots, a_n) = 1$.

Split toric varieties

This identity extends to the broader setting of split toric varieties. A variety X is a split toric variety if it contains a dense torus: a power of the multiplicative group $\mathbb{G}_m^d$, such that the group multiplication extends to a group action on X.

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$$x_1^{n_1} \dots x_k^{n_k} - x_1^{m_1} \dots x_k^{m_k} = 0,$$

such as products of projective spaces and projective bundles over $\mathbb{P}^1$ (e.g. Hirzebruch surfaces).

Multiplicities on toric varieties

Let $D_1, \ldots, D_n$ be the torus invariant prime divisors on X: the prime divisors in the compliment of the torus. On X, the rational points can be described using Cox coordinates $(a_1 : \cdots : a_n)$, similar to the homogeneous coordinates on projective space.

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For example, on $\mathbb{P}^1 \times \mathbb{P}^1$, the rational points are described by $((a : b), (c : d))$ with $\gcd(a, b) = \gcd(c, d) = 1$.
Using these coordinates, the multiplicity mult_p is once again simply the tuple of p-adic valuations.

M-points

Let $\mathfrak{M} \subset \mathbb{N}^n$ be a set containing the origin and write $M = ((D_1, \dots, D_n), \mathfrak{M})$. Then the set of *$M$-points* on (X, M) is

$$(X, M)(\mathbb{Z}) = \{P \in X(\mathbb{Q}) \mid \mathrm{mult}_p(P) \in \mathfrak{M} \text{ for all primes } p\}.$$

The notion of M-points extends the previously studied notions of Darmon points, Campana points and weak Campana points, which are all obtained by taking $\mathfrak{M}$ to be specific sets. We illustrate these notions when X is a split toric variety.

Darmon points and Campana points

Let $m_1, \ldots, m_n \in \mathbb{N} - \{0\}$.

- $\mathfrak{M} = \{(w_1, \ldots, w_n) \colon m_i | w_i\}$ gives the **Darmon points**

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$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon a_i \ m_i\text{-full}\}.$$

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(Here we recall an integer is m-ful if every prime dividing it appears with multiplicity $\geq m$.)
- $\mathfrak{M} = \{(w_1, \cdots : w_n) \colon \sum_{i=1}^n \frac{w_i}{m_i} \geq 1\} \cup \{(0, \ldots, 0)\}$ gives the **weak Campana points.** If $m_1, \ldots, m_n = m$, then
$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \prod_{i=1}^n a_i \ m\text{-full}\}.$$

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- $\mathfrak{M} = \{0,1\}^n$ gives "squarefree" points
  $(X,M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) : \ a_i \quad \text{squarefree}\}$.
- $\mathfrak{M} = \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ gives
  $(X,M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) : \ \prod_{i=1}^n a_i \quad \text{squarefree}\}$.

Heights

Now let us return to counting points. As before, consider an embedding $X \hookrightarrow \mathbb{P}^d$ corresponding to a line bundle L, and let H_L be the height on X obtained by restricting the Weil height

$$H(a_1 : \cdots : a_d) = \max(|a_1|, \ldots, |a_d|)$$

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to $X(\mathbb{Q})$.
Let $N_{(X,M),L}(B)$ be the number of M-points on (X, M) with height H_L at most B. For this quantity, we proved a generalization of Manin's conjecture for toric varieties.

Main theorem

We call a pair (X, M) *proper* if $\mathfrak{M}$ contains $d_1\mathbf{e}_1, \ldots, d_n\mathbf{e}_n$ for some positive integers $d_1, \ldots, d_n$ (satisfied for all examples).

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Theorem (B.M.,2025)

Let X be a smooth split toric variety and let $D_1, \ldots, D_n$ be torus invariant prime divisors. For every proper pair (X, M), there are explicit constants $a, b > 0$ such that

$$N_{(X,M),L}(B) = B^a(Q(\log B) + o(B^{-\theta})),$$

for some $\theta > 0$ and some polynomial Q of degree $b - 1$. In many cases, the leading coefficient c of Q is determined explicitly.

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The constants a, b and c admit geometric interpretations analogous to the invariants in Manin's conjecture, using a modified Picard group.

Earlier results

The only previously known cases were for Campana points (Pieropan-Schindler '24), and Darmon points (Shute-Streeter '24), for the specific choice of the log-anticanonical height: $L = \sum_{i=1}^{n} \frac{1}{m_i} D_i$.

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This theorem agrees with the Manin-type conjecture for Campana points as formulated by Pieropan, Smeets, Tanimoto and Várilly-Alvarado, as well as my extension to M-points.

Picard group of a pair

Let Γ_M be the (finite) set of minimal elements of $\mathfrak{M}$ in the partial order on $\mathbb{N}^n$. We define the groups of torus invariant divisors on X and (X, M) to be

$$\mathrm{Div}_T(X) = \bigoplus_{i=1}^{n} \mathbb{Z}[D_i],$$

and

$$\mathrm{Div}_T(X, M) = \bigoplus_{\mathbf{m} \in \Gamma_M} \mathbb{Z}[\tilde{D}_{\mathbf{m}}],$$

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and we consider the pullback map

$$\mathrm{pr}^* \colon \mathrm{Div}_T(X) \to \mathrm{Div}_T(X, M),$$

$$D_i \mapsto \sum_{\mathbf{m} \in \Gamma_M} m_i \tilde{D}_{\mathbf{m}}.$$

The effective cone

Define the *Picard group* $\mathrm{Pic}(X, M)$ to be the quotient of $\mathrm{Div}_T(X, M)$ by the image of principal divisors in $\mathrm{Div}_T(X)$, and let

$$\mathrm{Eff}(X, M) = \left\{ \sum_{\mathbf{m} \in \Gamma_M} a_{\mathbf{m}} \tilde{D}_{\mathbf{m}} : \ a_{\mathbf{m}} \geq 0 \right\} \subset \mathrm{Pic}(X, M) \otimes \mathbb{R}$$

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to be the canonical divisor of (X, M). Now

$$a = \min\{ t \in \mathbb{R} \colon \ t \, \mathrm{pr}^* L + K_{(X,M)} \in \mathrm{Eff}(X, M) \},$$

and b is the codimension of the minimal face of $\mathrm{Eff}(X, M)$ containing $a \, \mathrm{pr}^* L + K_{(X,M)}$.

The leading constant

Furthermore the leading coefficient c of Q is explicit, provided there exists an unique representative $D = a_1 D_1 + \cdots + a_n D_n$ of L such that $a \operatorname{pr}^* L + K_{(X,M)}$ is effective. In this case

$$c = \frac{\alpha}{a(b-1)!} C_\infty \prod_{p \text{ prime}} C_p,$$

where $\alpha, C_\infty \in \mathbb{Q}_{>0}$ are explicit volumes, and

$$C_p = \sum_{\mathbf{m} \in \mathfrak{M}} p^{-m_i a_i}.$$

Counting Darmon points and (weak) Campana points

If (X, M) is a pair corresponding to (weak) Campana points or Darmon points, then

$$a = \min\left\{ t \in \mathbb{R} \colon \; tL - \sum_{i=1}^{n} \frac{1}{m_i} D_i \in \mathrm{Eff}(X) \right\}.$$

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For Darmon and Campana points b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $A = aL - \sum_{i=1}^{n} \frac{1}{m_i} D_i$.

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For Darmon and Campana points b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $A = aL - \sum_{i=1}^{n} \frac{1}{m_i} D_i$. For weak Campana points, b is equal to this codimension plus

$$\#\left\{ \mathbf{w} \in \mathbb{N}^n \colon\ \sum_{i=1}^{n} \frac{w_i}{m_i} = 1, \quad \bigcap_{w_i > 0} D_i \neq \emptyset, \quad w_i = 0 \text{ if } D_i \subset \mathrm{Support(A)} \right\}$$

Examples

If $X = \mathbb{P}^{n-1}$ and we consider weak Campana points for $m_1, \dots, m_n = m$, then this gives that the number of primitive n-tuples $(a_1, \dots, a_n)$ with $\prod_{i=1}^{n} a_i$ m-ful and $\max(|a_1|, \dots, |a_n|) \leq B$ is asymptotic to $B^{n/m} Q(\log B)$, where Q has degree $\binom{m+n-1}{n-1} - \binom{m-1}{n-1} - n$.

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If $n = 3, m = 2$, then this becomes $B^{3/2} Q(\log B)$, where Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}} (1 - p^{-1})^6 \left(\frac{1 - p^{-3/2}}{(1 - p^{-1/2})^3} - 3p^{-1/2} \right) \approx 0.862.$$

Tauberian theorem

In order to prove the result, we use a multivariate Tauberian theorem by de la Brèteche.
Let $f : \mathbb{N}^n \to \mathbb{R}_{\geq 0}$ and consider the Dirichlet series

$$F(\mathbf{s}) = \sum \frac{f(r_1, \ldots, r_k)}{r_1^{s_1} \ldots r_k^{s_k}}.$$

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Suppose that there exists $\boldsymbol{\alpha} \in \mathbb{R}^r_{>0}$ such that F converges absolutely for all $\mathbf{s} > \boldsymbol{\alpha}$, and nonnegative linear forms $l_1, \ldots, l_{\tilde{n}}$ such that

$$H(\mathbf{s}) = F(\mathbf{s} + \boldsymbol{\alpha}) l_1(\mathbf{s}) \ldots l_{\tilde{n}}(\mathbf{s})$$

can be analytically continued to $\{\mathbf{s} \in \mathbb{C}^k \mid \mathrm{Re}(s_i) > -\delta\}$ for some $\delta > 0$. Additionally assume some bounds on the growth of F.

Tauberian theorem

Tauberian theorem (de la Brèteche, 2001)

Under these assumptions, we have

$$\sum_{r_1,\dots r_k \leq B} f(\mathbf{r}) = B^{\sum \alpha_i}(Q + o(B^{-\theta})),$$

where $\theta > 0$ and Q is a polynomial of degree at most $\tilde{n} - \operatorname{rank}(l_1, \dots, l_{\tilde{n}})$.

Tauberian theorem

Tauberian theorem (de la Brèteche, 2001)

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In our case, we use that $H_L(a_1 : \cdots : a_n) = \max(q_1, \dots, q_k)$, where $q_1, \dots, q_k$ are monomials in the coordinates and set $f(r_1, \dots, r_k)$ to be the number of $\mathbf{n} \in \mathbb{N}^n$ such that $q_i(\mathbf{n}) = r_i$ and $\mathrm{mult}_p(\mathbf{n}) \in \mathfrak{M}$.

Proof sketch using theorem

Using toric geometry, we show that there are $\boldsymbol{\alpha}$ and linear forms $\{l_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_M\}$ such that the assumptions are satisfied, $\sum_{i=1}^{k} \alpha_i = a$ and

$$\#\Gamma_M - \operatorname{rank}(\{l_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_M\}) = b - 1.$$

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$$\#\Gamma_M - \operatorname{rank}(\{l_{\mathbf{m}} \mid \mathbf{m} \in \Gamma_M\}) = b - 1.$$

This almost shows our theorem, but it does not exclude the possibility that Q has degree strictly smaller than $b-1$. A priory it could even be 0! To show it actually has degree $b-1$, we need to show that the $(b-1)$-th degree coefficient does not vanish.

Brèteche second theorem

For this we invoke a second theorem by de la Brèteche.

Theorem (de la Brèteche, 2001)

Under the assumptions of the previous theorem, additionally assume

1. $H(\mathbf{s})$ is a function of the linear forms,
2. $s_1 + \cdots + s_k$ is a strictly positive linear combination of the linear forms, and
3. $l_1(\alpha) = \cdots = l_{\tilde{n}}(\alpha) = 1$.

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Then $Q(\log B) \sim H(0,\ldots,0)B^{-\sum \alpha_i}\mathrm{Volume}(D(B))$, where

$$D(B) = \left\{ \mathbf{y} \in [0,\infty)^{\tilde{n}} \mid \prod_{i=1}^{\tilde{n}} y_i^{l_i(e_j)} \leq B \text{ for all } j = 1,\ldots,k \right\}.$$

End of the proof

If $a \operatorname{pr}^* L + K_{(X,M)}$ has an unique representation as an effective divisor, then this gives (after a lengthy computation) the leading coefficient in Q, proving the theorem under that assumption.

End of the proof

If $a\,\mathrm{pr}^* L + K_{(X,M)}$ has an unique representation as an effective divisor, then this gives (after a lengthy computation) the leading coefficient in Q, proving the theorem under that assumption.
If this is not the case, we can strategically pick a subset $\mathfrak{M}' \subset \mathfrak{M}$ such that the pair (X, M') has the same a- and b-invariants, and for which the above condition is satisfied. This gives a nonzero lower bound for the $(b-1)$-th degree coefficient and we are done.

Thank you for listening!

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