M-points of bounded height

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Manin's conjecture

In several previous talks we saw Manin's conjecture on the number of rational points of bounded height.
In this talk we will prove an extension of this by counting special kinds of rational points. For instance, the points with squarefree or squareful coordinates.

Split toric varieties

Let X be a smooth proper split toric variety. We can write

$$X = (\mathbb{A}^n \setminus V)/T,$$

where $T \subset \mathbb{G}_m^n$ is a subtorus and V is the vanishing locus of a collection of monomials.

Cox coordinates

Thus any rational point Q on X can be described using integer Cox coordinates

$$Q = (a_1 : \cdots : a_n),$$

where $(a_1, \ldots, a_n) \in (\mathbb{A}^n \setminus V)(\mathbb{Z})$ (i.e. a tuple of integers satisfying some GCD conditions).
Thus, for every prime number p, we have a multiplicity map

$$\mathrm{mult}_p \colon X(\mathbb{Q}) \to (\mathbb{N} \cup \{\infty\})^n$$

given by

$$(a_1 : \cdots : a_n) \mapsto (v_p(a_1), \ldots, v_p(a_n)).$$

M-points

Let $M \subset \mathbb{N}^n$ be a subset containing the origin. Then a rational point $Q \in X(\mathbb{Q})$ is an *M-point* if

$$\text{mult}_p(Q) \in M$$

for all prime numbers p. We denote the set of M-points by $(X, M)(\mathbb{Z})$.

Let us consider some examples. Take $m_1, \ldots, m_n$ to be positive integers.

Examples of M-points

1. $M = \{0,1\}^n$ gives
$$(X,M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \mid a_i \text{ squarefree}\}.$$
2. $M = \{(w_1,\ldots,w_n) \mid w_i = 0 \text{ or } \geq m_i\}$ gives the *Campana points*
$$(X,M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \mid a_i \text{ is } m_i\text{-ful}\}.$$
Here we recall that an integer n is m-ful if every prime number dividing it appears with multiplicity $\geq m$.
3. $M = \mathbf{0} \cup \{(w_1,w_2,w_3) \mid w_1 + w_2 + w_3 \geq 2\}$ gives
$$(X,M)(\mathbb{Z}) = \{(a_1 : a_2 : a_3) \mid a_1 a_2 a_3 \text{ is squareful}\}.$$

Heights

To study the distribution of M-points, we use heights.
For a big and nef line bundle L on X, $\Gamma(X, L)$ is generated by monomials $s_1, \ldots, s_k$ in the Cox coordinates, and the associated height is

$$H_L(a_1 : \cdots : a_n) = \max(|s_1(a_1, \ldots, a_n)|, \ldots, |s_k(a_1, \ldots, a_n)|).$$

Let

$$N_{(X,M),L}(B) = \#\{Q \in (X, M)(\mathbb{Z}) \mid H_L(Q) \leq B\}.$$

Asymptotic count

Our main result gives an explicit asymptotic for this counting function.

Theorem (B.M.,2025)

Let X be a smooth proper split toric variety over $\mathbb{Q}$, and let $L \in \mathrm{Pic}(X)$ be big and nef. Then there exists explicit constants $a, b > 0$ such that

$$N_{(X,M),L}(B) = B^a \left(Q\left(\log B\right) + O\left(B^{-\theta}\right)\right),$$

where Q is a polynomial of degree $b-1$ and $\theta > 0$. The leading coefficient of Q is also explicit, under some mild assumptions.

This is an extension of Manin's conjecture for toric varieties, and considerably generalizes the recent results of Pieropan and Schindler on Campana points.

The invariants a and b

The constants a, b are defined in terms of the Picard group $\mathrm{Pic}(X, M) \supset \mathrm{Pic}(X)$ and the canonical divisor $K_{(X,M)}$ of (X, M).

Example

The theorem has many novel implications. For example, the number of primitive integer triples $(x : y : z) \in \mathbb{P}^2(\mathbb{Q})$ with xyz squareful and $\max(x, y, z) \leq B$ tends to

$$cB^{3/2} \log(B)^3,$$

where

$$c = \prod_{p \text{ prime}} (1 - p^{-1})^6 \left(\frac{1 - p^{-3/2}}{(1 - p^{-1/2})^3} - 3p^{-1/2} \right) \approx 0.862.$$

Thank you for listening!

References:

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- M. Pieropan, A. Smeets, S. Tanimoto and A. Várilly-Alvarado. Campana points of bounded height on vector group compactifications. *Proc. Lond. Math. Soc. (3)*, 2021
- M. Pieropan and D. Schindler. Hyperbola method on toric varieties. *J. Éc. polytech. Math.*, 2024.