Counting special points of bounded height

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Overview

In this talk, I will give an introduction to Manin's conjecture on the number of rational points of bounded height.
After this, I will introduce an extension of Manin's conjecture to collections of special rational points (*M-points*), which I have proven for toric varieties. This will provide a framework to understand the distribution of special solutions of polynomial equations (squarefree, squareful, coprime etc).

Introduction

In arithmetic geometry, one central problem is to understand the rational solutions of a given system of polynomial equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0$$

over the rational numbers, i.e. the rational points on the corresponding algebraic variety.
For many varieties, the set of rational points will be infinite, such as for projective space or an elliptic curve of positive rank. In this case an interesting question is:
How many points smaller than a given "size" does the variety have?

Heights

This question is made precise using heights. On projective space $\mathbb{P}^{n-1}$, any rational point P can be written in homogeneous coordinates as $P = (a_1 : \cdots : a_n)$, where $a_1, \ldots, a_n$ are integers with no common divisor. In this notation, the *(Weil) height* of a rational point is given as

$$H(P) := \max(|a_1|, \ldots, |a_n|).$$

Heights

For any bound B, it is immediately clear that the number of rational points P with $H(P) \leq B$ is at most $(2B)^n$.In fact, we have an asymptotic

$$\#\{P \in \mathbb{P}^{n-1}(\mathbb{Q}) \mid H(P) \leq B\} \sim \frac{2^{n-1}}{\zeta(n)} B^n.$$

Heights

For a variety X with an embedding $i \colon X \hookrightarrow \mathbb{P}^{n-1}$ into projective space, we obtain a height H_L on the rational points of X by restricting the Weil height. The symbol L here denotes the divisor class corresponding to the embedding: it is the intersection of X with a hyperplane.

Counting points on varieties

Using the height H_L on X, we want to understand the asymptotics of the counting function

$$N_{X,L}(B) := \#\{P \in U(\mathbb{Q}) \mid H_L(P) \leq B\},$$

where U is an *open subset* of X.

Manin's conjecture

If X is rationally connected, meaning that any two points are connected by a rational curve ($\mathbb{P}^1$) then Manin's conjecture gives a prediction for the asymptotic of $N_{X,L}(B)$.
Many varieties are rationally connected, including toric varieties and Fano varieties such as hypersurfaces

$$\{f = 0\} \subset \mathbb{P}^d,$$

where f has degree at most d.

Manin's conjecture

We state a modern version of Manin's conjecture. Let $\mathrm{Eff}(X) \subset \mathrm{Pic}(X) \otimes \mathbb{R}$ be effective cone: the set of nonnegative linear combinations of divisors.

Manin's conjecture

Let X be a smooth projective rationally connected variety and let L be an ample line bundle on X. Then we have

$$N_{X,L}(B) \sim cB^a (\log B)^{b-1} \quad \text{as } B \to \infty,$$

where

$$a = \inf\{t \in \mathbb{R} \mid tL + K_X \in \mathrm{Eff}(X)\},$$

and b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $aL + K_X$.

Manin's conjecture

The constant c in the conjecture is also explicit but more subtle. While Manin's conjecture is still wide open, it is known for several varieties, such as toric varieties.

Theorem (Batyrev-Tschinkel, 1998)

Manin's conjecture is true for toric varieties, where the open set U is the dense torus.

We now illustrate this result with an example.

Example: product of projective lines

If we consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the Segre embedding into $\mathbb{P}^3$

$$((a:b),(c:d)) \mapsto (ac:ad:bc:bd),$$

then $\mathrm{Pic}(X) = \mathbb{Z}^2$ and $L = -\frac{1}{2}K_X$ so

$$N_{X,L}(B) \sim cB^2 \log B.$$

On the other hand, the embedding into $\mathbb{P}^5$ given by

$$((a:b),(c:d)) \mapsto (a^2c:a^2d:abc:abd:b^2c:b^2d)$$

gives

$$N_{X,L}(B) \sim c'B^2.$$

Rather than counting the full set of rational points, we can also count in interesting subsets. For instance, we would like to count the points on projective space for which the product of the coordinates is squareful. (An integer is squareful if it can be written as the product of a square and a cube.) In my thesis, I have studied such questions using the framework of M-points I introduced, expanding on the more restrictive notion of Campana points.
For instance, my results will imply that the number of triples $(x : y : z) \in \mathbb{P}^2(\mathbb{Z})$ such that xyz is squareful and $\max(|x|, |y|, |z|) \leq B$ is asymptotic to

$$cB^{3/2}(\log B)^3$$

as $B \to \infty$.

Multiplicities

M-points are defined using intersection multiplicities, which boils down to modular arithmetic. Let X be a projective variety with a subvariety D, both defined using equations over $\mathbb{Z}$. Then the *(intersection) multiplicity* of D with a point $Q \in X(\mathbb{Q})$ is the largest integer $n = n_p(D, Q)$ such that the reduction Q mod p^n lies on D. Thus for a given collection of prime divisors $D_1, \ldots, D_n$, we get for every prime number p a multiplicity map

$$\mathrm{mult}_p \colon X(\mathbb{Q}) \to (\mathbb{N} \cup \{\infty\})^n$$

$$Q \mapsto (n_p(D_1, Q), \ldots, n_p(D_n, Q)).$$

Thus for given divisors $D_1, \ldots, D_n$, we obtain a map

$$\mathrm{mult}_p \colon X(\mathbb{Q}) \to (\mathbb{N} \cup \{\infty\})^n,$$

given by the intersection multiplicities with the divisors.

M-points

For example, if $X = \mathbb{P}^{n-1}$ and $D_1, \ldots, D_n$ are the coordinate hyperplanes, then the multiplicity

$$\text{mult}_p(a_1 : \cdots : a_n) = (v_p(a_1), \ldots, v_p(a_n))$$

is simply the tuple of valuations. (Provided $\gcd(a_1, \ldots, a_n) = 1$.)
This identity also extends to split toric varieties. If $D_1, \ldots, D_n$ are the torus-invariant divisors on such a variety X, then points on X can be described using Cox coordinates $(a_1 : \cdots : a_n)$ (with a different scaling relation than on projective space), and the above identity still holds.
For example, on $\mathbb{P}^1 \times \mathbb{P}^1$, the Cox coordinates are the coordinates $((a : b), (c : d))$ with $\gcd(a, b) = \gcd(c, d) = 1$.

M-points

Let $\mathfrak{M} \subset \mathbb{N}^n$ be a set containing the origin and write $M = ((D_1, \ldots, D_n), \mathfrak{M})$. Then the set of *$M$-points* on (X, M) is

$$(X, M)(\mathbb{Z}) = \{P \in X(\mathbb{Q}) \mid \text{mult}_p(P) \in \mathfrak{M} \text{ for all primes } p\}.$$

The notion of M-points extends the previously studied notions of Darmon points, Campana points and weak Campana points, which are all obtained by taking $\mathfrak{M}$ to be specific sets. We illustrate these notions when X is projective space (or split toric).

Darmon points and Campana points

Let $m_1, \ldots, m_n \in \mathbb{N} - \{0\}$.

- $\mathfrak{M} = \{(w_1, \ldots, w_n) \colon m_i | w_i\}$ gives the **Darmon points**

$$(X, M)(\mathbb{Z}) = \{(\pm a_1^{m_1} : \cdots : \pm a_n^{m_n})\}.$$

- $\mathfrak{M} = \{(w_1, \ldots, w_n) \colon w_i = 0 \text{ or } w_i \geq m_i\}$ gives the **Campana points**

$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon a_i \ m_i\text{-full}\}.$$

(Here we recall an integer is m-ful if every prime dividing it appears with multiplicity $\geq m$.)

- $\mathfrak{M} = \{(w_1, \cdots : w_n) \colon \sum_{i=1}^{n} \frac{w_i}{m_i} \geq 1\} \cup \{(0, \ldots, 0\}$ gives the **weak Campana points.** If $m_1, \ldots, m_n = m$, then

$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \prod_{i=1}^{n} a_i \ m\text{-full}\}.$$

More examples of M-points

There are many more examples of M-points, such as

- $\mathfrak{M} = \{0,1\}^n$ gives "squarefree" points
  $(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \; a_i \quad \text{squarefree}\}$.
- $\mathfrak{M} = \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ gives
  $(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \; \prod_{i=1}^n a_i \quad \text{squarefree}\}$.

Heights

Now let us return to counting points. As before, consider an embedding $X \hookrightarrow \mathbb{P}^d$ corresponding to a line bundle L, and let H_L be the height on X obtained by restricting the Weil height

$$H(a_1 : \cdots : a_d) = \max(|a_1|, \ldots, |a_d|)$$

to $X(\mathbb{Q})$.
Let $N_{(X,M),L}(B)$ be the number of M-points on (X, M) with height H_L at most B. For this quantity, we proved a generalization of Manin's conjecture for toric varieties.

Main theorem

We call a pair (X, M) *proper* if $\mathfrak{M}$ contains $d_1\mathbf{e}_1, \dots, d_n\mathbf{e}_n$ for some positive integers $d_1, \dots, d_n$ (satisfied for all examples).

Theorem (B.M.,2025)

Let X be a smooth split toric variety and let $D_1, \dots, D_n$ be torus invariant prime divisors. For every proper pair (X, M), there are explicit constants $a, b > 0$ such that

$$N_{(X,M),L}(B) = B^a(Q(\log B) + o(B^{-\theta})),$$

for some $\theta > 0$ and some polynomial Q of degree $b - 1$. In many cases, the leading coefficient c of Q is determined explicitly.

The constants a, b and c admit geometric interpretations analogous to the invariants in Manin's conjecture, using a modified Picard group.

Earlier results

The only previously known cases were for Campana points (Pieropan-Schindler '24), and Darmon points (Shute-Streeter '24). Furthermore, both only treated the log-anticanonical height corresponding to $L = \sum_{i=1}^{n} \frac{1}{m_i} D_i$, and our result improves on their error term.
Furthermore, our theorem agrees with the prediction made for Campana points by Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

Examples

If $X = \mathbb{P}^{n-1}$ and we consider weak Campana points for $m_1, \dots, m_n = m$, then this gives that the number of primitive n-tuples $(a_1, \dots, a_n)$ with $\prod_{i=1}^n a_i$ m-ful and $\max(|a_1|, \dots, |a_n|) \leq B$ is asymptotic to $B^{n/m} Q(\log B)$, where Q has degree $\binom{m+n-1}{n-1} - \binom{m-1}{n-1} - n$.
If $n = 3, m = 2$, then this becomes $B^{3/2} Q(\log B)$, where Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}} (1 - p^{-1})^6 \left(\frac{1 - p^{-3/2}}{(1 - p^{-1/2})^3} - 3p^{-1/2} \right) \approx 0.862.$$

Picard group of a pair

Let Γ_M be the (finite) set of minimal elements of $\mathfrak{M}$ in the partial order on $\mathbb{N}^n$. We define the groups of torus invariant divisors on X and (X, M) to be

$$\mathrm{Div}_T(X) = \bigoplus_{i=1}^{n} \mathbb{Z}[D_i],$$

and

$$\mathrm{Div}_T(X, M) = \bigoplus_{\mathbf{m} \in \Gamma_M} \mathbb{Z}[\tilde{D}_{\mathbf{m}}],$$

and we consider the pullback map

$$\mathrm{pr}^* \colon \; \mathrm{Div}_T(X) \to \mathrm{Div}_T(X, M),$$

$$D_i \mapsto \sum_{\mathbf{m} \in \Gamma_M} m_i \tilde{D}_{\mathbf{m}}.$$

The effective cone

Define the *Picard group* $\operatorname{Pic}(X, M)$ to be the quotient of $\operatorname{Div}_T(X, M)$ by the image of principal divisors in $\operatorname{Div}_T(X)$, and let

$$\operatorname{Eff}(X, M) = \left\{ \sum_{\mathbf{m} \in \Gamma_M} a_{\mathbf{m}} \tilde{D}_{\mathbf{m}} \colon\ a_{\mathbf{m}} \geq 0 \right\} \subset \operatorname{Pic}(X, M) \otimes \mathbb{R}$$

be the *effective cone* of (X, M). Let $K_{(X,M)} = -\sum_{\mathbf{m} \in \Gamma_M} \tilde{D}_{\mathbf{m}}$ be the canonical divisor of (X, M). Now

$$a = \min\{t \in \mathbb{R} \colon\ t \operatorname{pr}^* L + K_{(X,M)} \in \operatorname{Eff}(X, M)\},$$

and b is the codimension of the minimal face of $\operatorname{Eff}(X, M)$ containing $a \operatorname{pr}^* L + K_{(X,M)}$.

Counting Darmon points and (weak) Campana points

If (X, M) is a pair corresponding to (weak) Campana points or Darmon points, then

$$a = \min \left\{ t \in \mathbb{R} \colon \; tL - \sum_{i=1}^{n} \tfrac{1}{m_i} D_i \in \mathrm{Eff}(X) \right\}.$$

For Darmon and Campana points b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $A = aL - \sum_{i=1}^{n} \frac{1}{m_i} D_i$. For weak Campana points, b is equal to this codimension plus

$$\# \left\{ \mathbf{w} \in \mathbb{N}^n \colon \; \sum_{i=1}^{n} \frac{w_i}{m_i} = 1, \quad \bigcap_{w_i > 0} D_i \neq \emptyset, \quad w_i = 0 \text{ if } D_i \subset \mathrm{Support(A)} \right\}$$

Thank you for listening!

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