M-points of bounded height

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Introduction

In this talk we will study the density of many special sets of rational points on algebraic varieties, using the framework of M-points.
On the projective plane, examples of sets of M-points are given by the sets of all $(x : y : z)$ such that

- x, y, z squarefree,
- x, y, z squareful,
- xyz squareful,
- $|xyz^2|$ a perfect cube,

and many more.

In this talk, we will introduce M-points and study the number of M-points of bounded height on split toric varieties, for which we will derive a precise asymptotic.
For instance, my results will imply that the number of triples $(x : y : z) \in \mathbb{P}^2(\mathbb{Z})$ such that xyz is squareful and $\max(|x|, |y|, |z|) \leq B$ is asymptotic to

$$cB^{3/2}(\log B)^3$$

as $B \to \infty$.

Multiplicities

Let X be a proper variety over $\mathbb{Q}$ (with a fixed model over $\mathbb{Z}$), and let D and P be a prime divisor and a point on X.
For every prime p we have an *intersection multiplicity* $n_p(D, P)$, which is the largest integer n such that P reduces to D modulo p^n (or ∞ if such an integer does not exist).
Thus for given divisors $D_1, \ldots, D_n$, we obtain a map

$$\mathrm{mult}_p \colon X(\mathbb{Q}) \to (\mathbb{N} \cup \{\infty\})^n,$$

given by the intersection multiplicities with the divisors.

M-points

Let $\mathfrak{M} \subset \mathbb{N}^n$ be a set containing the origin and write $M = ((D_1, \ldots, D_n), \mathfrak{M})$. Then the set of M-points on (X, M) is

$$(X, M)(\mathbb{Z}) = \{P \in X(\mathbb{Q}) \mid \text{mult}_p(P) \in \mathfrak{M} \text{ for all primes } p\}.$$

M-points on split toric varieties

Now let X be a smooth proper split toric variety, and let $D_1, \dots, D_n$ be the torus invariant prime divisors, such as $\mathbb{P}^{n-1}$ together with the coordinate hyperplanes. A point on X can be described using Cox coordinates

$$P = (a_1 : \cdots : a_n),$$

generalizing homogeneous coordinates on projective space.
We have

$$\mathrm{mult}_p(a_1 : \cdots : a_n) = (v_p(a_1), \dots, v_p(a_n)),$$

if the coordinates are integral and suitably coprime.

Darmon points and Campana points

We now give examples of M-points in these coordinates. Let $m_1, \ldots, m_n \in \mathbb{N} - \{0\}$.

- $\mathfrak{M} = \{(w_1, \ldots, w_n)\colon m_i | w_i\}$ gives the **Darmon points**
$$(X, M)(\mathbb{Z}) = \{(\pm a_1^{m_1} : \cdots : \pm a_n^{m_n})\}.$$
- $\mathfrak{M} = \{(w_1, \ldots, w_n)\colon w_i = 0 \text{ or } w_i \geq m_i\}$ gives the **Campana points**
$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n)\colon a_i \ m_i\text{-full}\}.$$
(Here we recall an integer is m-ful if every prime dividing it appears with multiplicity $\geq m$.)
- $\mathfrak{M} = \{(w_1, \cdots : w_n)\colon \sum_{i=1}^n \frac{w_i}{m_i} \geq 1\} \cup \{(0, \ldots, 0\}$ gives the **weak Campana points.** If $m_1, \ldots, m_n = m$, then
$$(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n)\colon \prod_{i=1}^n a_i \ m\text{-full}\}.$$

More examples of M-points

There are many more examples of M-points, such as

- $\mathfrak{M} = \{0, 1\}^n$ gives "squarefree" points
  $(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \ a_i \quad \text{squarefree}\}$.
- $\mathfrak{M} = \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ gives
  $(X, M)(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon \ \prod_{i=1}^n a_i \quad \text{squarefree}\}$.

Heights

How many M-points are there? Consider an embedding $X \hookrightarrow \mathbb{P}^d$ corresponding to a line bundle L, and let H_L be the height on X obtained by restricting the Weil height

$$H(x_1 : \cdots : x_d) = \max(|x_1|, \ldots, |x_d|)$$

to $X(\mathbb{Q})$.
Let $N_{(X,M),L}(B)$ be the number of M-points on (X, M) with height H_L at most B. For this quantity, we proved a generalization of Manin's conjecture for toric varieties.

Main theorem

We call a pair (X, M) *proper* if $\mathfrak{M}$ contains $d_1\mathbf{e}_1, \ldots, d_n\mathbf{e}_n$ for some positive integers $d_1, \ldots, d_n$ (satisfied for all examples).

Theorem (B.M.,2025)

Let X be a smooth split toric variety and let $D_1, \ldots, D_n$ be torus invariant prime divisors. For every proper pair (X, M), there are explicit constants $a, b > 0$ such that

$$N_{(X,M),L}(B) = B^a(Q(\log B) + o(B^{-\theta})),$$

for some $\theta > 0$ and some polynomial Q of degree $b - 1$. In many cases, the leading coefficient c of Q is determined explicitly.

The constants a, b and c admit geometric interpretations analogous to the invariants in Manin's conjecture.

Earlier results

The only previously known cases were for Campana points (Pieropan-Schindler '24), and Darmon points (Shute-Streeter '24). Furthermore, both only treated the log-anticanonical height corresponding to $L = \sum_{i=1}^{n} \frac{1}{m_i} D_i$, and our result improves on their error term.
Furthermore, our theorem agrees with the prediction made for Campana points by Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

Examples

If $X = \mathbb{P}^{n-1}$ and we consider weak Campana points for $m_1, \dots, m_n = m$, then this gives that the number of primitive n-tuples $(a_1, \dots, a_n)$ with $\prod_{i=1}^n a_i$ m-ful and $\max(|a_1|, \dots, |a_n|) \leq B$ is asymptotic to $B^{n/m} Q(\log B)$, where Q has degree $\binom{m+n-1}{n-1} - \binom{m-1}{n-1} - n$.
If $n = 3, m = 2$, then this becomes $B^{3/2} Q(\log B)$, where Q is a cubic polynomial with leading coefficient

$$\prod_{p \text{ prime}} (1 - p^{-1})^6 \left(\frac{1 - p^{-3/2}}{(1 - p^{-1/2})^3} - 3p^{-1/2} \right) \approx 0.862.$$

Picard group of a pair

Let Γ_M be the (finite) set of minimal elements of $\mathfrak{M}$ in the partial order on $\mathbb{N}^n$. We define the groups of torus invariant divisors on X and (X, M) to be

$$\mathrm{Div}_T(X) = \bigoplus_{i=1}^{n} \mathbb{Z}[D_i]$$

and

$$\mathrm{Div}_T(X, M) = \bigoplus_{\mathbf{m} \in \Gamma_M} \mathbb{Z}[\tilde{D}_{\mathbf{m}}],$$

and we consider the pullback map

$$\mathrm{pr}^*\colon \mathrm{Div}_T(X) \to \mathrm{Div}_T(X, M),$$

$$D_i \mapsto \sum_{\mathbf{m} \in \Gamma_M} m_i \tilde{D}_{\mathbf{m}}.$$

The effective cone

Define the *Picard group* $\mathrm{Pic}(X, M)$ to be the quotient of $\mathrm{Div}_T(X, M)$ by the image of principal divisors in $\mathrm{Div}_T(X)$, and let

$$\mathrm{Eff}(X, M) = \left\{ \sum_{\mathbf{m} \in \Gamma_M} a_{\mathbf{m}} \tilde{D}_{\mathbf{m}} : \ a_{\mathbf{m}} \geq 0 \right\} \subset \mathrm{Pic}(X, M) \otimes \mathbb{R}$$

be the *effective cone* of (X, M). Let $K_{(X,M)} = -\sum_{\mathbf{m} \in \Gamma_M} \tilde{D}_{\mathbf{m}}$ be the canonical divisor of (X, M). Now

$$a = \min\{t \in \mathbb{R} : \ t \, \mathrm{pr}^* L + K_{(X,M)} \in \mathrm{Eff}(X, M)\}$$

and b is the codimension of the minimal face of $\mathrm{Eff}(X, M)$ containing $a \, \mathrm{pr}^* L + K_{(X,M)}$.

Counting Darmon points and (weak) Campana points

If (X, M) is a pair corresponding to (weak) Campana points or Darmon points, then

$$a = \min \left\{ t \in \mathbb{R} \colon \; tL - \sum_{i=1}^{n} \tfrac{1}{m_i} D_i \in \mathrm{Eff}(X) \right\}.$$

For Darmon and Campana points b is the codimension of the minimal face of $\mathrm{Eff}(X)$ containing $A = aL - \sum_{i=1}^{n} \frac{1}{m_i} D_i$. For weak Campana points, b is equal to this codimension plus

$$\# \left\{ \mathbf{w} \in \mathbb{N}^n \colon \; \sum_{i=1}^{n} \frac{w_i}{m_i} = 1, \quad \bigcap_{w_i > 0} D_i \neq \emptyset, \quad w_i = 0 \text{ if } D_i \subset \mathrm{Support(A)} \right\}$$

Thank you for listening!

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