M-points and adelic approximation on toric varieties

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19 November 2024

Introduction

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When can local points be lifted to global points?

For example, when can a solution of an equation over $\mathbb{Z}/n\mathbb{Z}$ be lifted to a solution in $\mathbb{Z}$?
The archtypical result in this topic is the Chinese remainder theorem.

Chinese remainder theorem

The Chinese remainder theorem is equivalent to the statement that the map

$$\mathbb{Z} \to \prod_{i=1}^{k} \mathbb{Z}/p_i^n\mathbb{Z}$$

is surjective for all $n \in \mathbb{N}$ and distinct primes p_i.

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$$\mathbb{Z} \to \prod_{i=1}^{k} \mathbb{Z}/p_i^n\mathbb{Z}$$

is surjective for all $n \in \mathbb{N}$ and distinct primes p_i. Using the p-adics, this can be restated more elegantly:

Chinese remainder theorem

The diagonal embedding

$$\mathbb{Z} \hookrightarrow \prod_{p \text{ prime}} \mathbb{Z}_p$$

has dense image.

Geometric analogue

This has a natural geometric counterpart:

Chinese remainder theorem

The diagonal embedding

$$\mathbb{C}[t] \hookrightarrow \prod_{p \in \mathbb{C}} \mathbb{C}[\![t-p]\!]$$

has dense image.

Which means that for $p_1, \dots, p_n \in \mathbb{C}$ we can find a polynomial f having any desired values and derivatives at those points. **(Include image.)**

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These extensions lead to the notions of *weak* and *strong approximation*, and have been extensively studied over the last century. In this talk we will extend these notions by imposing further arithmetic conditions (squarefreeness, coprimality, squareful, etc) on the points. This leads to the new notion of M-approximation.

Extensions

In this talk I will characterize when M-approximation holds on (split) toric varieties, such as projective space. This will allow us to tackle questions such as:
Are the squarefree integers dense in the product of the "squarefree p-adics"?
As well as similar questions related to valuations.

Preliminaries

In this talk, we work over a fixed **PF field** K. Such a field is either

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- a prime ideal in $\mathcal{O}_K$ if K is a number field, or
- a closed point in C if K is a function field.

If K is a number field, then an **infinite place** v is an embedding $v \colon K \to \mathbb{C}$, up to conjugation, while a function field has no infinite places. We denote the set of places of K by Ω_K.

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Preliminaries

For any place v, let K_v be the completion of K at v. If v is finite, we let $\mathcal{O}_v$ be the ring of elements with nonnegative valuation.

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Let X be a proper (e.g. projective) variety over K. An **integral model** of X is a proper scheme $\mathcal{X}$ over $\mathcal{O}_K$ or C such that $\mathcal{X}_K = X$. Such a model can be found by "clearing the denominators in the equations defining X". For example, $\mathbb{P}^n_{\mathbb{Z}}$ is an integral model of $\mathbb{P}^n_{\mathbb{Q}}$.

As $\mathcal{X}$ is proper, any K-point lifts to an unique $\mathcal{O}_S$-point. Similarly, K_v-points lift to $\mathcal{O}_v$-points for every finite place v. We will use this to define special kinds of rational points, called M-points.

Local intersection multiplicity

Let D be a prime divisor on X with closure $\mathcal{D}$ in $\mathcal{X}$. Let v be a finite place and let $P \in X(K_v)$.
The **local intersection multiplicity** $n_v(D, P)$ of $\mathcal{D}$ and P at v is defined as the maximal $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ such that P reduces to $\mathcal{D}$ modulo π_v^n, where π_v is a uniformizer of $\mathcal{O}_v$ (such as p in $\mathbb{Z}_p$).

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If $\{f = 0\}$ is a local equation of $\mathcal{D}$ around P, then it follows that $n_v(\mathcal{D}, P) = v(f(P))$ is the valuation of the value of f at P.

Geometric interpretation

In the function field setting, this agrees with the local intersection multiplicity between curves and divisors. (**Include picture intersections here.**)

M-points

Let $D_1, \ldots, D_n$ be divisors on X and let $\mathfrak{M} \subset \overline{\mathbb{N}}^n$ be a subset containing $(0, \ldots, 0)$. Set $M = ((D_1, \ldots, D_n), \mathfrak{M})$. For a finite place v let

$$\mathrm{mult}_v \colon X(K_v) \to \overline{\mathbb{N}}^n$$

be the map given by

$$P \mapsto (n_v(D_1, P), \ldots, n_v(D_n, P)).$$

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Then a point $P \in X(K_v)$ an **v-adic M-point** if $\mathrm{mult}_v(P) \in \mathfrak{M}$. Similarly, an point $P \in X(K)$ is an **M-point over $\mathcal{O}_S$** if $\mathrm{mult}_v(P) \in \mathfrak{M}$ for all $v \in \Omega_K \setminus S$.

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Then a point $P \in X(K_v)$ an *v*-**adic** *M*-**point** if $\mathrm{mult}_v(P) \in \mathfrak{M}$.
Similarly, an point $P \in X(K)$ is an *M*-**point over** $\mathcal{O}_S$ if $\mathrm{mult}_v(P) \in \mathfrak{M}$ for all $v \in \Omega_K \setminus S$.
We denote the set of v-adic M-points by $(\mathcal{X}, M)(\mathcal{O}_v)$ and the set of M-points over $\mathcal{O}_S$ by $(\mathcal{X}, M)(\mathcal{O}_S)$.

Example: integral points

If $\mathfrak{M} = \{0\}^k \times \overline{\mathbb{N}}^{n-k}$, then

$$(\mathcal{X}, \mathcal{M})(\mathcal{O}_S) = \mathcal{U}(\mathcal{O}_S)$$

are the $\mathcal{O}_S$-integral points on the open $\mathcal{U} = \mathcal{X} \setminus (\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k) \subset \mathcal{X}$.

Example: M-points on projective space

If we take $X = \mathbb{P}^{n-1}_K$ and the divisors to be the coordinate hyperplanes, then for any finite place v and a point $P = (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}(K_v)$, the multiplicity map is simply $\mathrm{mult}_v(P) = (v(x_1), \ldots, v(x_n))$, if the x_i are taken to be coprime.

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Examples on projective space I

For example,

1. if $\mathfrak{M} = \{0\}^n$, then

$$(\mathbb{P}^{n-1}, M)(\mathcal{O}_S) = \{(x_1 : \cdots : x_n) | x_i \in \mathcal{O}_S^\times\} \cong (\mathcal{O}_S^\times)^{n-1}$$

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3. if $\mathfrak{M}$ consists of the elements in $\mathbb{N}^n$ with at most a single coordinate nonzero, then
$$(\mathbb{P}^{n-1}, M)(\mathbb{Z}) = \{(x_1 : \cdots : x_n) | x_i \in \mathbb{Z}, \gcd(x_i, x_j) = 1 \, \forall i \neq j\},$$
as the points do not reduce to the intersection of two divisors modulo any prime.

Special cases

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2. If $\mathfrak{M} = \mathbb{N}_{\geq m_1} \times \cdots \times \mathbb{N}_{\geq m_n}$, then the M points on $(\mathcal{X}, M)$ are called **Campana points** (for the given weights $m_1, \ldots, m_n$).

Examples on projective space II

Again, let $\mathcal{X} = \mathbb{P}^{n-1}$ and consider the points over $\mathbb{Z}$.

1. The squarefree points are

$$(\mathbb{P}^{n-1}, M)(\mathbb{Z}) = \{(x_1 : \cdots : x_n) \mid x_i \text{ squarefree}\}.$$

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$$(\mathbb{P}^{n-1}, M)(\mathbb{Z}) = \{(x_1 : \cdots : x_n) \mid x_i \text{ is } m_i\text{-full}\},$$
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Similar descriptions exist for other rings (with trivial class group), by replacing the role of $\pm$ with the units in the ring.

For simplicity, we now assume every element in $\mathfrak{M}$ has all coordinates finite. We also set $U = X \setminus (D_1 \cup \cdots \cup D_n)$.

Adelic M-points

Let us now return to the problem considered at the start. In order to generalize the Chinese remainder theorem, we need an analogue of $\prod_{p \text{ prime}} \mathbb{Z}_p$.

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Definition: integral adelic M-points

Let $T \subset S$, the space of **integral adelic M-points** over S prime to T to be

$$(\mathcal{X}, M)(\mathbf{A}_{\mathcal{O}_S}^T) := \prod_{v \in \Omega_K \setminus S} (\mathcal{X}, M)(\mathcal{O}_v) \times \prod_{v \in S \setminus T} U(K_v).$$

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Using this language, we see that $(\mathbb{P}^1, M)(\mathbf{A}_{\mathbb{Z}}^T) = \prod_{p \text{ prime}} \mathbb{Z}_p$, where $M = ((0:1), \{0\})$ and $T = \{\infty\}$ consists of the infinite place.

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This space is independent of the choice of the integral model $\mathcal{X}$ and contains $U(K)$ as a subset.

M-approximation

Now we can finally define M-approximation.

Definition: M-approximation

Let $T \subset \Omega_K$ be a finite set of places. The pair (X, M) satisfies M-**approximation off** T if the diagonal embedding

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This can be rephrased as follows: given a finite set of places S containing T and a point $P_v \in U(K_v)$ for all $v \in S \setminus T$, then there exist points $P \in (\mathcal{X}, M)(\mathcal{O}_S)$ which approximate all P_v arbitrarily well.

Special cases

If $n = 0$, so no conditions are imposed, then $(X, M)(\mathbf{A}_K^T) = X(\mathbf{A}_K^T)$ are the usual adeles on X and M-approximation is the same as *weak approximation off* T.

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If (X, M) is the pair for Campana points, then M-approximation is the same as weak Campana approximation as considered by Nakahara and Streeter (2024).

Intermezzo: split toric varieties

All statements about projective space thus far naturally extend to (split) toric varieties, which are varieties X containing a torus $\mathbb{G}_m^d$ as a dense open subspace, such that the action of the torus on itself extends to X. For the remainder of the talk, X will be a smooth toric variety.

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$$(x : y : z : w) = (\lambda x : \mu y : \lambda z : \lambda^r \mu w),$$

for all units μ, λ.

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In particular, $\mathcal{H}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{H}_1 = \mathrm{Bl}_{(0:0:1)} \mathbb{P}^2$.

Divisors and rays

Using the coordinates on a toric variety we choose $D_i := \{X_i = 0\}$ to be the vanishing locus of the i-th coordinate. (This way, the $D_1, \dots, D_n$ are exactly the torus invariant prime divisors.)

Divisors and rays

Using the coordinates on a toric variety we choose $D_i := \{X_i = 0\}$ to be the vanishing locus of the i-th coordinate. (This way, the $D_1, \ldots, D_n$ are exactly the torus invariant prime divisors.) A toric variety corresponds to a combinatorial object called a fan, and this associates to every D_i the ray generator: a vector $u_i \in \mathbb{Z}^d$, where d is the dimension of X.
For example, for $\mathbb{P}^{n-1}$, these are
$u_1 = \mathbf{e}_1, \ldots, u_{n-1} = \mathbf{e}_{n-1}, u_n = -\sum_{i=1}^n \mathbf{e}_i$. **(Add picture)**
For $\mathcal{H}_r$, these are
$u_1 = (-1, r), u_2 = (0, 1), u_3 = (1, 0), u_4 = (0, -1)$.

The lattice N_M

Consider the surjective linear map

$$\phi\colon \mathbb{N}^n \to \mathbb{Z}^d$$

defined by $\mathbf{e}_i \mapsto u_i$. The image of $\mathfrak{M} \subset \mathbb{N}^n$ in $\mathbb{Z}^d$ generates a monoid N_M^+ and a lattice N_M by considering (nonnegative) linear combinations.

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These two invariants control much of the arithmetic of (X, M) and $\mathbb{Z}^d / N_M$ can be viewed as some kind of fundamental group.

M-approximation for toric varieties

Theorem (B. Moerman), 2024

Let X be a smooth toric variety over a PF field K with $D_1, \ldots, D_n$ the torus invariant prime divisors. Let T be a nonempty finite set of places.

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Here $\rho(K) = \{1\}$ for global fields and "most" other fields. On the other hand $\rho(K) = \mathbb{N} \setminus char(K)\mathbb{N}$ if K is a function field of a curve over an algebraically closed field.

M-approximation for toric varieties

Theorem (B. Moerman), 2024

Let X be a smooth toric variety over a PF field K with $D_1, \ldots, D_n$ the torus invariant prime divisors. Let T be a nonempty finite set of places. Then

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($\rho(K)$ is described in general in my preprint.)

Consequences: integral points

We summarize a variety of consequences.

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The squarefree points and the Campana points on a smooth toric variety both satisfy M-approximation over any PF field and any weights $m_1, \ldots, m_n$.

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Proof.

Since ϕ is surjective, it follows that $N_M^+ = N_M$. Since both $m_i u_i$ and $(m_i + 1)u_i$ lie in the image for all i, $N_M^+ = \mathbb{Z}^d$. □

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The Darmon points on a smooth toric variety over a number field satisfy M-approximation if and only if there exist no finite covers $Y \to X$ of integral varieties only ramified over $D_1, \ldots, D_n$ with ramification indices $r_i | m_i$.

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The only case previously known were for curves, by work of Christensen (2020) and Santens (2023).

Proof idea in general

Then we construct the point P approximating the v-adic points by letting the i-th coordinate of P be a product of powers of such prime elements, and we use the

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$$P_i = (u_{i,1} p_i^{v_i,1} : \dots : u_{i,n} p_i^{v_{i,n}}) \in \mathbb{P}^{n-1}(\mathbb{Q}_{p_i}),$$

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where $u_{i,j} \in \mathbb{Z}_{p_i}^\times$.By multiplying everything (coordinate-wise) by

$$\prod_{i=1}^{s} (p_i^{-v_{i,1}} : \dots : p_i^{-v_{i,n}}),$$

we can assume that $P_i \in \mathbb{G}_m^d(\mathbb{Z}_{p_i})$.

Proof sketch

Now choose $\mathbf{m}_1, \ldots, \mathbf{m}_l \in \mathfrak{M}$ whose images generate $\mathbb{Z}^d$. These $(\mathbf{m}_1, \ldots, \mathbf{m}_l)$ give a surjective map $\mathbb{Z}^l \to \mathbb{Z}^d$, and thus induce a surjective map $(\mathbb{Z}_{p_i}^\times)^l \to (\mathbb{Z}_{p_i}^\times)^d$. By combining this with Dirichlet's theorem on arithmetic progressions, we can construct such P explicitly.

Proof sketch (details)

To be precise: for each $1, \ldots, s$ want $C_{\mathbf{m}_1,i}, \ldots, C_{\mathbf{m}_l,i} \in \mathbb{Z}_{p_i}^{\times}$ with

$$\prod_{k=1}^{l} (C_{\mathbf{m}_1,i}^{m_{k,1}}, \ldots, C_{\mathbf{m}_k,i}^{m_{k,n}}) = t(u_{i,1}, \ldots, u_{i,n}),$$

for some $t \in \mathbb{Z}_{p_i}^{\times}$. By the surjectivity, such $C_{\mathbf{m}_j,i}$ exist.

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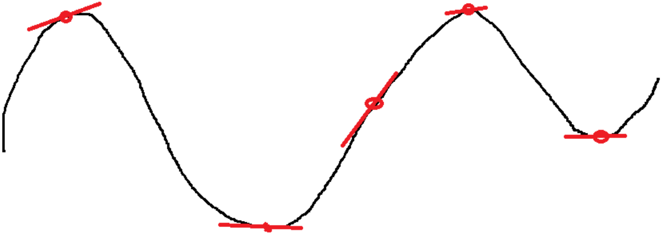
We also have some limited control over $|p - x_{v_0}|_{v_0}$, which is necessary for the first part of the theorem. This is a generalization of Dirichlet's theorem on arithmetic progressions, but it requires more sophisticated tools than Chebotarev's density theorem (if $K \neq \mathbb{Q}$), as that gives little control over the infinite places.

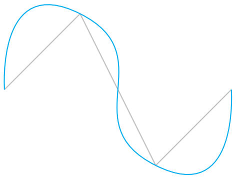
Proof idea in general: other fields

For global function fields, we prove an analogous result using similar methods. Over function fields over an infinite field, we prove a similar result using genericity arguments akin to Bertini's theorem/generic smoothness.

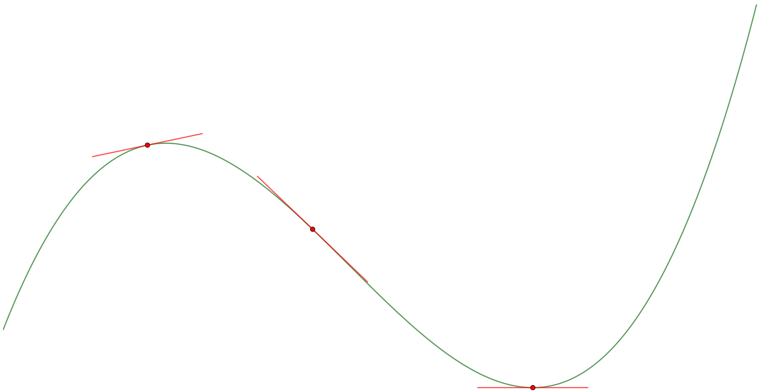
Test



Test 2



Test3



Test3