Approximation of generalized Campana points

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Conventions

For simplicity, we work over $\mathbb{Q}$ and $\mathbb{Z}$, but the results work more generally over number fields and function fields of curves over any field. We denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Many types of points

Let X be a variety with integral model $\mathcal{X}$ and let $D_1, \ldots, D_n$ be divisors on X with Zariski closure $\mathcal{D}_1, \ldots, \mathcal{D}_n$. There are a lot of special subsets of rational points defined relative to these, such as

- integral points,
- Campana points and weak Campana points,
- Darmon points etc.

We introduce W-points as a common framework for these points.

Multiplicity map

For a prime p we define the multiplicity map

$$\mathrm{mult}_p\colon X(\mathbb{Q}_p) \to \overline{\mathbb{N}}^n$$

$$P \mapsto (n_p(P, \mathcal{D}_1), \ldots, n_p(P, \mathcal{D}_n)).$$

$\mathcal{W}$-points

Given $\mathfrak{W} \subset \overline{\mathbb{N}}^n$ containing $\{0, \ldots, 0\}$ we set $\mathcal{W} = ((\mathcal{D}_1, \ldots, \mathcal{D}_n), \mathfrak{W})$ and we define the set of *p-adic $\mathcal{W}$-points* as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}_p) = \{P \in X \mid \mathrm{mult}_p(P) \in W\},$$

and the set of *$\mathcal{W}$-points over $\mathbb{Z}$* as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{P \in X \mid \mathrm{mult}_p(P) \in W \text{ for all primes } p\}.$$

Multiplicities on toric varieties

We take X to be a complete smooth split toric variety, with integral model $\mathcal{X}$ induced by the fan. We let $D_1, \ldots, D_n$ be the torus-invariant prime divisors (on projective space: coordinate hyperplanes).

Multiplicities on toric varieties

We can represent a point P on $X(\mathbb{Q})$ by its Cox coordinates $(a_1, \dots, a_n)$ (corresponding to the D_i) and by scaling we can assume that $a_i \in \mathbb{Z}$, and that for every prime p there exists a cone σ such that $p \nmid a_i$ for all i not corresponding to the cone. Then we have

$$\mathrm{mult}_p(P) = (v_p(a_1), \dots, v_p(a_n)).$$

Examples of $\mathcal{W}$-points

Let $m_1, \ldots, m_n \in \overline{\mathbb{N}} - \{0\}$

- $\mathfrak{W} = \{0,1\}^n$ gives "squarefree" points $(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(a_1, \ldots, a_n)\colon a_i \text{ squarefree}\}$.
- $\mathfrak{W} = \{(w_1, \ldots, w_n)\colon m_i | w_i\}$ gives the Darmon points

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(\pm a_1^{m_1}, \ldots, \pm a_n^{m_n})\}$$

- $\mathfrak{W} = \{(w_1, \ldots, w_n)\colon w_i = 0 \text{ or } w_i \geq m_i\}$ gives the Campana points

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(a_1, \ldots, a_n)\colon a_i \ m_i\text{-full}\}.$$

W-approximation

Now we generalize strong approximation. We say that $\mathcal{X}$ satisfies *(integral) $\mathcal{W}$-approximation* if the map

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathcal{X}, \mathcal{W})(\mathbb{Z}_p) \times X(\mathbb{R})$$

has dense image,
and say it satisfies *(integral) $\mathcal{W}$-approximation* off ∞ if the map

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathcal{X}, \mathcal{W})(\mathbb{Z}_p).$$

This property implies that $(\mathcal{X}, \mathcal{W})(\mathbb{Z})$ is not thin.

W-approximation for toric varieties

When does this happen in the toric setup? Consider the fan of X in $\mathbb{Z}^d$, $d = \dim X$. Then we get a homomorphism

$$\mathbb{N}^n \to \mathbb{Z}^d$$

sending $e_i \mapsto n_i$, where n_i is the ray generator associated to D_i. (For $\mathbb{P}^d$ we take $n_i = e_i$ if $i \leq d$ and $n_{i+1} = -\sum_{i=1}^d e_i$.) Using this map, W generates a submonoid

$$N_W^+ \subset \mathbb{Z}^d$$

and a subgroup

$$N_W \subset \mathbb{Z}^d.$$

W-approximation for toric varieties

Theorem (B.M.)

1. $\mathcal{X}$ satisfies $\mathcal{W}$-approximation off ∞ if and only if $N_W = \mathbb{Z}^d$.
2. $\mathcal{X}$ satisfies $\mathcal{W}$-approximation if and only if $N_W^+ = \mathbb{Z}^d$

$\mathcal{W}$-approx for Darmon points

Let W give the Darmon points as before and write Y for the associated stack. Then we have an isomorphism

$$\pi_1^{\acute{e}t}(Y_{\mathbb{C}}) \cong \widehat{\mathbb{Z}^n / N_W}.$$

Together with the theorem this gives

Corollary (B.M.)

X satisfies $\mathcal{W}$-approximation off ∞ if and only if $Y_{\mathbb{C}}$ is simply connected. Furthermore, X satisfies $\mathcal{W}$-approximation if and only if furthermore all global sections of $Y_{\mathbb{C}}$ are constant.

In particular, if we take $m_i \in \{1, \infty\}$, we recover conditions for strong approximation for split toric varieties.

As an example, if $X = \mathbb{P}^n$, then $Y_{\mathbb{C}}$ is simply connected if and only if $\gcd(m_i, m_j) = 1$ for all $i \neq j$.

$\mathcal{W}$-approx for Campana points

Let W give the Campana points on (X, Δ) as before. We have

Corollary (B.M.)

X satisfies $\mathcal{W}$-approximation (off ∞) if and only if it is satisfied for $(X, \lfloor \Delta \rfloor)$. In particular it holds if $m_i \neq \infty$ for any i.

In general

The results transfer verbatim to number fields, and after slight modification also for function fields of curves.