Approximation of generalized Campana points

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Outline

The p-adics

In number theory interested in equations mod p^n.
Define **valuation**

$$v_p(p^n \cdot \tfrac{a}{b}) := n \text{ if } p \nmid a, b, \quad v_p(0) = \infty.$$

We have $p^n | x$ iff $v_p(x) \geq n$. Induces absolute value

$$|a|_p := p^{-v_p(a)}.$$

Complete $\mathbb{Q}$ with $|\cdot|_p$ to get $\mathbb{Q}_p$, which is a locally compact field.
The closure of $\mathbb{Z}$ is the ring

$$\mathbb{Z}_p = \{a \in \mathbb{Q}_p \colon |a|_p \leq 1\}.$$

The p-adics

Can write for $x \in \mathbb{Q}_p$

$$x = \sum_{i=k}^{\infty} a_i p^i, \quad k \in \mathbb{Z}, a_i \in \{0, \ldots, p-1\},$$

with $x \in \mathbb{Z}_p$ if $k \geq 0$. (So we have $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$.)

Strong approximation

By definition, $\mathbb{Z}$ dense in $\mathbb{Z}_p$. By the Chinese remainder theorem, if $p \neq q$ primes, then

$$x \equiv a \bmod p^n, \quad x \equiv b \bmod q^m$$

is equivalent to

$$x \equiv c \bmod p^n q^m$$

for some specific c. Therefore the diagonal map

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z}_p \times \mathbb{Z}_q \\ x &\mapsto (x, x) \end{aligned}$$

has dense image.

Strong approximation

In the same way prove

Strong approximation theorem (weak form)

The diagonal embedding

$$\mathbb{Z} \to \prod_{p \text{ prime}} \mathbb{Z}_p$$

has dense image.

Generalizations of strong approximation

Main result generalizes strong approximation in 3 main ways:

1. Generalizing $\mathbb{Q}$ to different fields.
2. Considering different spaces.
3. Restricting to subsets of $\mathbb{Z}$.

Points 1 & 2 have been studied extensively, but very little is known about point 3.

For simplicity restrict the field to $\mathbb{Q}$ and the space to projective space.

Projective space

Let K be a field, we define **projective space** of dimension n as

$$\mathbb{P}^n(K) = (K^{n+1} \setminus \{0, \dots, 0\})/K^\times.$$

So in coordinates: $(\lambda a_0 : \cdots : \lambda a_n) = (a_0 : \cdots : a_n)$.
If K has a topology, then $\mathbb{P}^n(K)$ is given the quotient topology.

Multiplicities

Any $P \in \mathbb{P}^n(\mathbb{Q})$ can be given integer, coprime coordinates.
Similarly $P \in \mathbb{P}^n(\mathbb{Q}_p)$ can be written as $(a_0 : \cdots : a_n)$ with $a_i \in \mathbb{Z}_p$ and $v_p(a_i) = 0$ for some i.
As this is unique up to $\mathbb{Z}_p^\times$, get a map

$$\begin{aligned} \text{mult}\colon \mathbb{P}^n(\mathbb{Q}_p) &\to \overline{\mathbb{N}}^{n+1} \\ (a_0 : \cdots : a_n) &\mapsto (v_p(a_0), \ldots, v_p(a_n)). \end{aligned}$$

Here $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

W-points

For a subset $W \subset \overline{\mathbb{N}}^{n+1}$ containing $\{(0, \ldots, 0)\}$ we define the subspace of **p-adic W-points**

$$(\mathbb{P}^n, W)(\mathbb{Z}_p) := \{P \in \mathbb{P}^n(\mathbb{Q}_p) \colon \operatorname{mult}_p(P) \in W\},$$

and the set of **W-points over $\mathbb{Z}$**

$$(\mathbb{P}^n, W)(\mathbb{Z}) := \{P \in \mathbb{P}^n(\mathbb{Q}) \colon \operatorname{mult}_p(P) \in W \text{ for all primes } p\}.$$

For example, if $W = \{(0, \ldots, 0)\}$, then
$(\mathbb{P}^n, W)(\mathbb{Z}) = \{(\pm 1 : \cdots : \pm 1)\}$ and
$(\mathbb{P}^n, W)(\mathbb{Z}_p) = \{(\mathbb{Z}_p^\times : \cdots : \mathbb{Z}_p^\times)\}$

Examples of W-points

- $n = 1$, $W = \overline{\mathbb{N}} \times \{0\}$ give $(\mathbb{P}^1, W)(\mathbb{Z}_p) = \{(a : 1) \colon a \in \mathbb{Z}_p\} \cong \mathbb{Z}_p$ and similarly for $\mathbb{Z}$.
- $W = \{0, 1\}^{n+1}$ gives $(\mathbb{P}^n, W)(\mathbb{Z}) = \{(a_0 : \cdots : a_n) \colon a_i \text{ squarefree}\}$.

Examples of W-points

Let $m_0, \ldots, m_n$ be positive integers. Take coordinates to be coprime.

- If $W = \{(w_0, \ldots, w_n)\colon m_i | w_i\}$, then

$$(\mathbb{P}^n, W)(\mathbb{Z}) = \{(\pm a_0^{m_0} : \cdots : \pm a_n^{m_n})\}$$

  are the **Darmon points**.

- If $W = \{(w_0, \ldots, w_n)\colon w_i = 0 \text{ or } w_i \geq m_i\}$, then

$$(\mathbb{P}^n, W)(\mathbb{Z}) = \{(a_0 : \cdots : a_n)\colon a_i \ m_i\text{-full}\}$$

  are the **Campana points**.
  (An integer a is m-full if $p|a$ implies $p^m|a$.)

W-approximation

$\mathbb{P}^n$ satisfies W-**approximation** if

$$(\mathbb{P}^n, W)(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathbb{P}^n, W)(\mathbb{Z}_p) \times \mathbb{P}^n(\mathbb{R})$$

has dense image.

$\mathbb{P}^n$ satisfies W-**approximation off** $\mathbb{R}$ if

$$(\mathbb{P}^n, W)(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathbb{P}^n, W)(\mathbb{Z}_p)$$

has dense image.

To understand when this holds for $W \subset \mathbb{N}^{n+1}$, define linear map

$$\phi\colon \mathbb{N}^{n+1} \to \mathbb{Z}^n$$

by

$$e_i \mapsto \begin{cases} e_i & \text{for } i > 0 \\ -(e_1 + \cdots + e_n) & \text{for } i = 0. \end{cases}$$

W-approximation

Set $W' = \{(w_0, \ldots, w_n) \in W \colon w_i = 0 \text{ for some } i\}$.

Theorem

$\mathbb{P}^n$ satisfies W-approximation off $\mathbb{R}$ if and only if the integral linear span of $\phi(W')$ is $\mathbb{Z}^n$.
Furthermore $\mathbb{P}^n$ satisfies W-approximation if and only if the integer coefficients in the linear combinations can be chosen nonnegative.

W-approximation

Corollary

$\mathbb{P}^n$ always satisfies W-approximation for Campana points.

Corollary

$\mathbb{P}^n$ satisfies W-approximation for Darmon points if and only if $\gcd(m_0, \ldots, m_n) = 1$. (We recall $W = \{(w_0, \ldots, w_n)\colon m_i | w_i\}$)

Example: For Darmon points with $m_0 = 2$, $m_1 = 3$ this gives: for primes $p_1, \ldots, p_k$ and $Q_i = (c_i : d_i) \in \mathbb{P}^1(\mathbb{Z}/p_i\mathbb{Z})$ and $Q = (c : d) \in \mathbb{P}^1(\mathbb{R})$, can find $a, b \in \mathbb{Z}$ such that $P = (a^2 : b^3)$ reduces to Q_i and approximates Q well.

W-approximation

Example: For Darmon points with $m_0 = m_1 = 2$ we have that W-approximation (off $\mathbb{R}$) is not satisfied since 2 mod 5 is not of the form $\pm a^2$ mod 5, so no W-point reduces to $(2:1)$ modulo 5.

Example: taking $W = \mathbb{N} \times \{0\}$ gives original strong approximation theorem. ($\mathbb{Z} \to \prod_p \mathbb{Z}_p$ dense.)

The full result is much more general, and works over toric varieties and number fields, as well as function fields.